Singular BSDEs and PDEs Arising in Optimal Liquidation Problems

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To my family

Abstract

This dissertation analyzes BSDEs and PDEs with singular terminal condition arising in models of optimal portfolio liquidation. Portfolio liquidation problems have received considerable attention in the financial mathematics literature in recent years. Their main characteristic is the singular terminal condition of the value function induced by the liquidation constraint, which translates into a singular terminal state constraint on the associated BSDE or PDE.

The dissertation consists of three chapters. The first chapter analyzes a multi-asset portfolio liquidation problem with instantaneous and persistent price impact and stochastic resilience. We show that the value function can be described by a multi-dimensional BSRDE with a singular terminal condition. We prove the existence of a solution to this BSRDE and show that it can be approximated by a sequence of the solutions to BSRDEs with finite increasing terminal condition. A novel a priori estimate for the approximating BSRDEs is established for the verification argument.

The second chapter considers a portfolio liquidation problem with unbounded cost coefficients. We establish the existence of a unique nonnegative continuous viscosity solution to the HJB equation. The existence result is based on a novel comparison principle for semi-continuous viscosity sub-/supersolutions for singular PDEs. Continuity of the viscosity solution is enough to carry out the verification argument.

The third chapter studies an optimal liquidation problem under ambiguity with respect to price impact parameters. In this case the value function can be characterized by the solution to a semilinear PDE with superlinear gradient. We first prove the existence of a solution in the viscosity sense by extending our comparison principle for singular PDEs. Higher regularity is then established using an asymptotic expansion of the solution at the terminal time.

Zusammenfassung

Diese Dissertation analysiert BSDEs und PDEs mit singulären Endbedingungen, welche in Problemen der optimalen Portfolioliquidierung auftreten. In den vergangenen Jahren haben Portfolioliquidierungsprobleme in der Literatur zur Finanzmathematik große Aufmerksamkeit erhalten. Ihre wichtigste Eigenschaft ist die singuläre Endbedingung der durch die Liquidierungsbedingung induzierten Wertfunktion, welche eine singuläre Endbedingung der zugehörigen BSDE oder PDE impliziert.

Diese Arbeit besteht aus drei Kapiteln. Das erste Kapitel analysiert ein Portfolioliquidierungsproblem für mehrere Wertpapiere mit sofortigem und anhaltendem Preiseinfluss und stochastischer Resilienz. Wir zeigen, dass die Wertfunktion durch eine mehrdimensionale BSRDE mit singulärer Endbedingung beschrieben werden kann. Wir weisen die Existenz einer Lösung dieser BSRDE nach und zeigen, dass diese durch eine Folge von Lösungen von BSRDEs mit endlicher und wachsender Endbedingung approximiert werden kann. Eine neue a priori-Abschätzung für die approximierenden BSRDEs wird für den Nachweis hergeleitet.

Das zweite Kapitel betrachtet ein Portfolioliquidierungsproblem mit unbeschränkten Kostenkoeffizienten. Wir weisen die Existenz einer eindeutigen nichtnegativen Viskositätslösung der HJB-Gleichung nach. Das Existenzresultat basiert auf einem neuartigen Vergleichsprinzip für semi-stetige Viskositätssub-/-superlösungen für singuläre PDEs. Stetigkeit der Viskositätslösung ist hinreichend für das Verifikationsargument.

Im dritten Kapitel untersuchen wir ein optimales Liquidierungsproblem unter Mehrdeutigkeit der Parameter des Preiseinflusses. In diesem Fall kann die Wertfunktion durch die Lösung einer semilinearen PDE mit superlinearem Gradienten beschrieben werden. Zuerst zeigen wir die Existenz einer Viskositätslösung indem wir unser Vergleichsprinzip für singuläre PDEs erweitern. Sodann weisen wir die Regularität mit einer asymptotischen Entwicklung der Lösung am Endzeitpunkt nach.

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Contents

Notation

I. Sets
• $\mathbb{R}^d$ denotes the d-dimensional Euclidian space.
• $\mathcal{S}^d$ is the set of symmetric $d \times d$ matrices and $\mathcal{S}^d_+$ is the set of nonnegative definite matrices in $\mathcal{S}^d$. For any two matrices A, B from $\mathcal{S}^d$ we write $A > B$ and $A \geq B$ if $A - B$ is positive definite, respectively nonnegative definite.
• I_d denotes the $d \times d$ identity matrix.
• $|A| := \sqrt{\sum_{ij} a_{ij}^2}$ denotes the norm of a vector or matrix $A = (a_{ij})$. For any matrix $B \in \mathcal{S}^d$, the largest (smallest) eigenvalue is denoted by $b_{\max}$ ($b_{\min}$) and $|B|_{2,2} := b_{\max}$ denotes the induced matrix norm.

II. Functional spaces
• $C_b(\mathbb{R}^d)$, $C_b(I \times \mathbb{R}^d)$ are the spaces of bounded continuous functions on $\mathbb{R}^d$, respectively, on $I \times \mathbb{R}^d$. Here, I is a compact subset of $\mathbb{R}$.
• $C_m(\mathbb{R}^d)$ (resp. $C_m(I \times \mathbb{R}^d)$) is the set of all continuous functions ϕ satisfying that

$$\psi := \frac{\phi(y)}{1 + |y|^m} \in C_b(\mathbb{R}^d)(\text{resp. } \psi := \frac{\phi(t,y)}{1 + |y|^m} \in C_b(I \times \mathbb{R}^d)).$$

• $USC_m(I \times \mathbb{R}^d)$ (resp. $LSC_m(I \times \mathbb{R}^d)$) is the set of all functions ϕ that have at most polynomial growth of order m in the second variable uniformly with respect to $t \in I$ and are upper (resp. lower) semi-continuous on $I \times \mathbb{R}^d$.
• $\mathcal{SSG}^{\pm}_m(I \times \mathbb{R}^d)$ is the set of all functions ϕ satisfying that

$$\liminf_{|y| \to \infty} \frac{\pm\phi(t,y)}{|y|^m} \geq 0, \text{ uniformly with respect to } t \in I.$$

III. Integration and probability
• $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space.
• $Q \ll P$ means that the measure Q is absolutely continuous with respect to the measure P.
• $L^q_{\mathcal{F}}(0,T;\mathbb{R}^d)$ is the space of all adapted $\mathbb{R}^d$-valued processes $(f_t)_{t \in [0,T]}$ satisfying that $\mathbb{E}[\int_0^T |f_t|^q \, dt)]^{1/q} < \infty$; $L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^d)$ is the space of all essentially bounded stochastic processes. Here, $T \in (0, \infty)$
• $\mathcal{S}^q_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^d))$ is the space of all adapted processes with continuous paths satisfying that $\mathbb{E}[\sup_{t \in [0,T]} |f_t|^q]^{1/q} < \infty$; $\mathcal{S}^\infty_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^d))$ is the space of all essentially bounded stochastic processes with continuous paths.
• $H^q_{\mathcal{F}}(0,T;\mathbb{R}^d)$ is the space of all the adapted processes $(f_t)_{t \in [0,T]}$ satisfying that $\mathbb{E}[(\int_0^T |f_t|^2 \, dt)^{q/2}]^{1/q} < \infty$.

- We say that a sequence of stochastic processes $\{f^n(\cdot)\}_{n\in\mathbb{N}}$ converges compactly to $f(\cdot)$ on $[0,T)$ if $\sup_{t\in I}|f_n(t)-f(t)|$ converges to 0 in the $\mathbb{P}$-a.s. sense on every compact subinterval I.

IV. Notational conventions

- For any $y\in\mathbb{R}^d$, we deonte

$$\langle y\rangle := (1+|y|^2)^{1/2}.$$

- Whenever the notation T^- appears we mean that the statement holds for all $T'<T$ when T^- is replaced by T', e.g.

$$L^2_{\mathcal{F}}(0,T^-;\mathbb{R}^d) = \bigcap_{T'<T} L^2_{\mathcal{F}}(0,T';\mathbb{R}^d).$$

- The operator D denotes the gradient with respect to the space variable.
- All equations and inequalities are to be understood in the $\mathbb{P}$-a.s. sense.
- We adopt the convention that C is a constant that may vary from line to line.

1. Introduction

Traditional financial market models assume that price fluctuations follow some exogenous stochastic process and that arbitrarily large positions of assets can be traded at the current market price without affecting this price. Empirical evidence indicates that large trades, however, are often settled at 'worse' prices than small trades due to adverse market impact. Market impact models have long been studied in the economics literature; see Kyle[Kyl85], Easley and O' Hara[EO87] and references therein. The focus of the economics literature on market impact is typically on the role of information asymmetries and how these asymmetries affect asset prices. Recently, in the wake of the dramatic increase in automation trading, problems of optimal execution of large trades have also received considerable attention in the financial mathematics literature. While the focus of the economics literature is on deriving *endogenous* impact functions from information asymmetries, this line of models focuses on structural models within which to derive optimal portfolio strategies for *endogenously* given impact functions. In a model of optimal portfolio liquidation, a financial trader needs to unwind a large asset portfolio within a given time period. In this thesis we consider novel stochastic control problems arising in models of optimal portfolio liquidation.

The first papers dealing with optimal liquidation problems in the financial mathematics literature were those of Bertsimas and Lo [BL98] and Almgren and Chriss [AC01]. Two kinds of price impact were distinguished in their papers. The temporary (or instantaneous) impact depends only on the present trading rate and does not affect future trades; the permanent impact adds an extra drift to the price dynamic and does affect future trades. For linear temporary price impact and linear permanent impact, Bertsimas and Lo [BL98] derived dynamic optimal trading strategies for a risk-neutral investor based on the minimization of the expected cost of execution. Almgren and Chriss [AC01] extended this model to risk-averse investors and gave a closed-form solution for the optimal execution strategy in a mean-variance framework. Huberman and Stanzl [HS04] showed that the linear functions are the only choice of the *permanent* price impact for which the model is free from arbitrage. The choice of the *temporary* impact function is more flexible. For instance, Almgren [Alm03] assumed that the magnitude of the temporary market impact is a power law function of the trading rate, which was estimated through a square-root law in [Alm03] and a 3/5 power law in [ATHL05] based on the available historical transaction data. To better capture the intertemporal nature of supply and demand in the market, Obizhaeva and Wang [OW13] proposed another kind of price impact that is persistent (or transient) with the impact of past trades on current prices decaying over time.

1.1. Mathematical background

The main characteristic of optimal liquidation problems is the liquidation constraint at the end of the trading period. The terminal constraint induces a singularity of the value function at the terminal time. Most of the early research on optimal execution problems focused on deterministic market impact functions. These models were often solved by using calculus of variation techniques where the liquidation constraint causes no mathematical difficulties. When stochastic market impact functions are allowed, the calculus of variation technique usually can not be applied. Instead, one has to solve the resulting stochastic control problems either via Bellman's dynamic programming principle or by using the stochastic maximum principle. In both cases, the liquidation constraint causes significant mathematical challenges, as it induces a *singular terminal condition* of the Hamilton-Jacobi-Bellman (HJB) equation or the adjoint equation in the stochastic maximum principle.

When linear temporary price impact and quadratic risk terms are considered, and only absolutely continuous trading strategies are admissible, the linear dynamics of the portfolio process suggests a quadratic ansatz for the value function. Depending on the dynamics of the cost coefficients, the HJB equation reduces to a one-dimensional or multi-dimensional ordinary differential equation, a partial differential equation (in a Markovian setting), or a backward stochastic differential equation, a backward stochastic partial differential equation (in a non-Markovian setting) with singular terminal value. Solving optimal liquidation problems under model uncertainty naturally leads to a class of the singular HJB equations whose driver has a superlinear growth in the gradient. Without model uncertainty, the driver is independent of the gradient.

1.1.1. The penalization approach

The most common approach to overcome the mathematical challenge resulting from the terminal singularity of the HJB equation is based on a penalization method. The idea is to approximate the solution to the HJB equation with singular terminal condition by a sequence of the solutions to HJB equations with finite increasing terminal condition, from which a minimal solution to the singular HJB equation can then be derived. Popier [Pop06, Pop07] applied the penalization approach to solve a singular BSDE and to obtain minimal solutions in different settings. Later, in a non-Markovian optimal liquidation problem, Ankirchner et al. [AJK14] showed that the value function can be charaterized in terms of a minimal solution to a singular BSDE. This model was generalized to allow for both active and passive orders by Kruse and Popier [KP16], who solved the control problem by establishing the existence of a minimal supersolution to a singular BSDE with jumps. Graewe et al. [GHQ15] investigated a mixed Markov/non-Markov liquidation problem by analyzing the minimal solution to a singular BSPDE, which was extended to the case of degenerate parabolic equation in [HQZ16]. Popier and Zhou [PZ19] analyzed the optimal liquidation problem under drift and volatility uncertainty in a non-Markovian setting and characterized the value function by the minimal supersolution of a second-order BSDE with monotone generator and singular terminal condition.

In Chapter 2, which is based on [HX19], we study a multi-asset portfolio liquidation problem with instantaneous and persistent price impact and stochastic resilience using the penalization method. In the case of multi-asset portfolios, additional difficulties arise when employing the penalization method. In addition to the convergence of the value function, the convergence of the optimal trading strategies is required for the verification argument. This can already be observed in Kratz and Schöneborn [KS15] where a multi-asset Almgren-Chriss model with dark pools was considered. They derived a minimal solution to a coupled ODE system with singular terminal condition and established a priori estimates of the (suitably weighted) solutions to the approximating ODE systems in diagonal form. This particular form of estimates allowed them to infer the convergence of the optimal trading strategies for the unconstrained models to an admissible liquidation strategy for the original problem. In the one-dimensional setting, much coarser a priori estimates are sufficient to carry out the verification. In our model, the HJB equation reduces to a multi-dimensional backward stochastic Riccati differential equation (BSRDE) with a singular terminal condition in one component. We establish a novel a priori estimate for the approximating BSRDEs, from which we deduce that the value function can indeed be described by the singular BSRDE. As a byproduct we obtain a convergence result for the single-asset model analyzed in [GH17].

1.1.2. The asymptotic approach

An alternative approach based on an asymptotic expansion to solve the HJB equations with singular terminal values was introduced by Graewe et al. [GHS18] and later extended in [GH17]. The key of this approach is to determine the precise asymptotic behavior of a potential solution to the HJB equation at the terminal time. It was shown in [GH17, GHS18] that the asymptotics of the solution educate an asymptotic ansatz that reduces the HJB equation with singular terminal value to a BSDE or PDE with a finite terminal condition yet singular driver. A similar idea has previously been used in [AK12] where they established the existence of a unique viscosity solution to the singular HJB equation with a constant temporary price impact coefficent. Using this asymptotic approach, Graewe et al. [GHS18] proved the existence of a smooth solution to the singular HJB equation with *bounded* coefficients. In Chapter 3, which is based on [HX18], we establish the existence of a unique nonnegative continuous viscosity solution to the singular HJB equation with possibly *unbounded* coefficients. The proof is based on a novel comparison principle for semi-continuous viscosity sub- and supersolutions for PDEs with singular terminal value. Continuity of the viscosity solution is enough to carry out the verification argument.

In Chapter 4, which is based on [HXZ19], we study the portfolio liquidation problem considered in Chapter 3 when the investor is uncertain about the factor dynamics driving trading costs. We prove that the value function to our control problem can be characterized by the solution to a semilinear PDE with superlinear gradient, monotone generator and singular terminal value. Our idea is to first obtain a viscosity solution by extending the comparison principle in Chapter 3 and then to deduce a higher regularity of this solution by applying the asymptotic approach. We also establish an asymptotic analysis of the robust model for small amounts of uncertainty and analyze the effect of robustness on optimal trading strategies and liquidation costs. In particular, in our model factor uncertainty is observationally equivalent to increased risk aversion. This suggests that factor uncertainty increases liquidation rates.

1.2. Summary of Chapter 2

In this Chapter, we consider a multi-asset portfolio liquidation problem with instantaneous and persistent price impact and stochastic resilience. This problem leads to a stochastic control problem in the form

$$\operatorname*{ess\,inf}_{\xi\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)} \mathbb{E}\left[\int_0^T \Big(\frac{1}{2}\xi(s)^{\mathrm{T}}\Lambda\xi(s)+Y(s)^{\mathrm{T}}\xi(s)+\frac{1}{2}X(s)^{\mathrm{T}}\Sigma(s)X(s)\Big)\,ds\right]$$

subject to the state dynamics

$$\begin{cases} X(s)=x-\displaystyle\int_0^s \xi(r)\,dr, \quad s\in[0,T],\\ X(T)=0,\\ Y(s)=y+\displaystyle\int_0^s \big(-\rho(r)Y(r)+\gamma\xi(r)\big)\,dr, \quad s\in[0,T]. \end{cases}$$

Here, Λ is a deterministic positive definite matrix that describes an instantaneous impact factor as in [AC01]. The process Σ is a progressively measurable essentially bounded and nonnegative definite matrix that describes the volatility of portfolios holding. The coefficient γ is a diagonal matrix and the process Y describes the persistent price impacts caused by past trades in block-shaped limit order book markets with constant order book depths $\frac{1}{\gamma_i}>0$ for the various asset as in [OW13]. The process ρ is a progressively measurable essentially bounded and nonnegative definite diagonal matrix that describes how fast the order books recover from past trades.

Several multi-dimensional liquidation models with *deterministic* cost functions and *deterministic* resilience have previously been considered in the literature. The special case $\rho\equiv 0, y=0$, and $\Sigma\equiv const.$ corresponds to the multiple-asset model of Almgren and Chriss [AC01]. This model was generalized by Kratz and Schöneborn

[KS14] to discrete-time multi-asset liquidation problem when an investor trades simultaneously in a traditional venue and a dark pool. In the follow-up work [KS15], the same authors studied a continuous-time multi-asset liquidation problem with dark pools. The benchmark case of *deterministic* coefficients and *zero persistent impact* ($Y = 0$) corresponds to the model in [KS15] without a dark pool. A model of optimal basket liquidation for a CARA investor with general deterministic cost function was analyzed by Schied et al [SST10]. Later, Schöneborn [Sch16] considered an infinite-horizon multi-asset portfolio liquidation problem for a von Neumann-Morgenstern investor with general deterministic temporary and linear permanent impact functions. Alfonsi et al. [AKS16] considered a discrete-time model of optimal basket liquidation with linear transient price impact and general deterministic resilience. In a continuous-time version of [AKS16], Schneider and Lillo [SL18] derived theoretical limits for the size and form of cross-impact that can be directly verified on data from the condition of absence of dynamic arbitrage.

In our model, the value function can be described by the matrix-valued BSRDE

$$\begin{aligned}-dQ(t) = &\Bigg(- Q(t)\begin{pmatrix}-I_d\\ \gamma\end{pmatrix}\Lambda^{-1}\begin{pmatrix}-I_d & \gamma\end{pmatrix}Q(t) + Q(t)\begin{pmatrix}0 & 0\\ 0 & -\rho(t)\end{pmatrix}\\ &+\begin{pmatrix}0 & 0\\ 0 & -\rho(t)\end{pmatrix}Q(t) + \begin{pmatrix}\Sigma(t) & 0\\ 0 & \gamma^{-1}\rho(t)+\rho(t)\gamma^{-1}\end{pmatrix}\Bigg)\,dt\\ &- M(t)\,dW(t), \quad t\in[0,T)\end{aligned}$$

with a singular terminal condition

$$\liminf_{t\to T}|Q(t)| = +\infty.$$

We first analyze the unconstrained problems with finite end costs and show that the value functions for unconstrained problems are given by the solutions to BSRDE systems with finite terminal value by using verification argument for linear quadratic optimal control problem given in [Bis76, Pen92, KT03]. For the benchmark case of uncorrelated assets the system of BSRDEs can be decomposed into a series of subsystems for which a priori estimates similar to those in [GH17] can be established. Then we prove that the solutions to the BSRDE systems can be uniformly bounded from above and below on compact time interval by two benchmark models with uncorrelated assets. This allows us to prove that the pointwise (in time) limit of the solutions to these unconstrained systems exists when the degree of penalization tends to infinity. This limit yields a candidate value function for the liquidation problem.

The verification argument is much more involved. It requires a much finer a priori estimate for the approximating BSRDE systems, from which we can prove the convergence of the optimal trading strategies and to carry out the verification argument. We extend the ideas in [KS15] to optimal liquidation models with stochastic resilience. Due to the presence of the persistent impact factor Y, our estimates are much more complicated. In particular, our BSRDE system has a first order term that requires additional estimates before the desired estimates of the (suitably weighted) solution can indeed be established in diagonal form. Moreover, our optimal portfolio process is given in terms of nonhomogeneous differential equations, which cannot be solved directly by simply multiplying $\sqrt{\Lambda}$ as in [KS15].

1.3. Summary of Chapter 3

In this Chapter, we consider a portfolio liquidation problem under price-sensitive market impact. Precisely, we analyze the following stochastic control problem:

$$\operatorname*{ess\,inf}_{\xi,\mu} E\left[\int_0^T \eta(Y_s)|\xi_s|^2 + \theta\gamma(Y_s)|\mu_s|^2 + \lambda(Y_s)|X_s^{\xi,\mu}|^2\,ds\right] \tag{1.1}$$

subject to the state dynamics

$$\begin{aligned} dY_t &= b(Y_t)dt + \sigma(Y_t)dW_t, \quad Y_0 = y \\ dX_t^{\xi,\mu} &= -\xi_t\,dt - \mu_t\,dN_t, \quad X_0^{\xi,\mu} = x \end{aligned} \tag{1.2}$$

and the terminal state constraint

$$X_T^{\xi,\mu} = 0. \tag{1.3}$$

where N is a Poisson process and W is a $\tilde{d}$-dimensional standard Brownian motion, which is independent of N. We assume that the cost coefficients η, λ, γ are continuous and of polynomial growth, that η is twice continuously differentiable and that the diffusion coefficients b, σ are Lipschitz continuous.

Control problems of the form (1.1)-(1.3) arise in models of optimal portfolio liquidation under market impact when a trader can simultaneously trade in a primary venue and a dark pool. Dark pools are alternative trading venues that allow investors to reduce market impact and hence trading costs by submitting liquidity that is shielded from public view. Trade execution is uncertain, though, as trades will be settled only if matching liquidity becomes available. In such models, $X^{\xi,\mu}$ describes the portfolio process when the trader submits orders at rate ξ to the primary venue for immediate execution and orders of size μ to the dark pool. Dark pool execution is governed by the Poisson process N. The process Y denotes a factor process that drives trading costs. The process η describes the instantaneous market impact; it often describes the so-called market depth. The process γ describes adverse selection costs associated with dark pool trading while λ usually describes market risk, e.g. the volatility of a portfolio holding.

We show that the corresponding HJB equation reduces to the following singular terminal value problem:

$$\begin{cases} -\partial_t v(t,y) - \mathcal{L}v(t,y) - F(y, v(t,y)) = 0, & (t,y) \in [0,T) \times \mathbb{R}^d, \\ \lim\limits_{t\to T} v(t,y) = +\infty & \text{locally uniformly on } \mathbb{R}^d, \end{cases}$$

where the nonlinearity F is given by

$$F(y,v) := \lambda(y) - \frac{|v|^2}{\eta(y)} + \frac{\theta\gamma(y)v}{\gamma(y)+|v|} - \theta v.$$

We establish the existence of a unique nonnegative continuous viscosity solution to this PDE with possibly unbounded cost coefficients. We show that the existence of a *continuous* viscosity solution is enough to carry out the verification arguments and to give a representation of the optimal control in feedback form. As a by-product, we obtain that the minimal nonnegative solution to the stochastic HJB equation in [AJK14] is indeed the unique nonnegative solution to their singular BSDE with unbounded coefficients. This complements the analysis in [KP16, GHS18].

Existence of *continuous* solutions to HJB equations associated with control problems of the form (1.1)-(1.3) has so far mostly been established under L^∞ assumptions on the model parameters by many authors. For instance, the existence of unique continuous viscosity solution was established when η is a constant and λ is of polynomial growth in [AK12]. Existence and uniqueness of solutions in suitable Sobolev spaces for bounded stochastic cost and diffusion coefficients was proved in [GHQ15, HQZ16]; classical solutions were considered in [GHS18]. The restriction to constant market impact terms and/or bounded impact functions and diffusion coefficients seems unsatisfactory. The framework in [Sch13a] allows for unbounded coefficients but requires strong a priori estimates on the market impact term that are not satisfied in our main example. Complementing the analysis in [Sch13a] our results show when value function derived in terms of Dawson-Watson superprocesses therein solves the HJB equation in the viscosity sense.

Due to the singular terminal state constraint, the standard comparison principles for PDEs cannot be applied. Instead, we prove a novel comparison principle, which shows that if some form of asymptotic dominance holds at the terminal time, then dominance holds near the terminal time. Subsequently, we construct smooth sub- and supersolutions that satisfy the required asymptotic dominance condition. This allows us to apply Perron's method to establish an upper semi-continuous subsolution and a lower semi-continuous supersolution that are bounded from above/below by the smooth solutions. From this, we infer that the semi-continuous solutions can be applied to the comparison principle, which then implies the existence of the desired continuous viscosity solution.

1.4. Summary of Chapter 4

We study a portfolio liquidation problem when the investor is uncertain about the factor dynamics driving trading costs. Specifically, we consider the stochastic control problem

$$\inf_{\xi} \sup_{Q\in\mathcal{Q}} \left(\mathbb{E}_Q\left[\int_0^T \eta(Y_s)|\xi_s|^p + \lambda(Y_s)|X_s|^p\, ds\right] - \Upsilon(Q)\right)$$

subject to the state dynamics

$$\begin{aligned} dY_t &= b(Y_t)dt + \sigma(Y_t)dW_t, \quad Y_0 = y \\ dX_t &= -\xi_t \, dt, \quad X_0 = x \end{aligned}$$

and the terminal state constraint

$$X_T = 0,$$

where $\mathcal{Q}$ is a set of probability measures that are absolutely continuous with respect to a benchmark measure $\mathbb{P}$. Instead of restricting the set of probability measures ex ante, we add a penalty term $\Upsilon(Q)$ to the objective function. The benchmark case where $\mathcal{Q}$ contains a single element has been analyzed in [GHS18] and Chapter 3 with dark pools.

Only few papers have studied the optimal liquidation problem under model uncertainty. Nyström et al. [NAZ14] and Cartea et al. [CJ17, CDJ17] considered problems of optimal liquidation with *limit orders* for a CARA investor who is uncertain about both the drift and the volatility of the underlying reference price process, respectively for a risk-neutral investor who is uncertain about the arrival rate of market orders, the fill probability of limit orders and the dynamics of the asset price. In these papers strict liquidation is not required. Lorenz and Schied [LS13] studied the drift dependence of optimal trade execution strategies under transient price impact with exponential resilience and strict liquidation constraint. Later, Schied [Sch13b] analyzed the impact on optimal trading strategies with respect to misspecification of the law of the unaffected price process in a model which only allows instantaneous price impact. Bismuth et al. [BGP19] considered a portfolio liquidation model for a CARA investor that is uncertain about the drift of the reference price process but did not require a strict liquidation constraint. All three papers focused on misspecification of the reference price process but did not consider the resulting robust control problem. Moreover, they assumed that the market impact parameters are known. Our model is different; we analyze the effect of uncertainty about the model parameters.

Popier and Zhou [PZ19] analyzed the optimal liquidation problem under drift and volatility uncertainty in a non-Markovian setting, while we focus on the drift uncertainty about the factor model. In the spirit of convex risk measure theory, we add a penalty term to the cost function. We also obtain much stronger regularity properties of the value function which allows us to study the effect of uncertainty on optimal trading strategies and costs in greater detail.

Under a suitable scaling property on the penalty function (corresponding to homothetic preferences) that had first been introduced by Maenhout [Mae04], we prove that the value function to our control problem can be characterized by the solution to a semi-linear PDE with superlinear gradient, monotone generator and singular terminal value

$$\begin{cases} -\partial_t v(t,y) - \mathcal{L}v(t,y) - H(y, Dv(t,y)) - F(y, v(t,y)) = 0, & (t,y) \in [0,T) \times \mathbb{R}^d, \\ \lim\limits_{t\to T} v(t,y) = +\infty, & \text{locally uniformly on } \mathbb{R}^d. \end{cases}$$

where

$$F(y,v) := \lambda(y) - \frac{|v|^{\beta+1}}{\beta\eta(y)^\beta}, \quad H(y, Dv) := \theta^\alpha |\sigma^*(y)Dv|^{\alpha+1}.$$

Our first main contribution is to prove that this PDE admits a unique nonnegative viscosity solution of polynomial growth under standard assumptions on the factor process and the cost coefficients by extending the comparison principle considered in Chapter 3. The dependence of the generator on the gradient requires additional regularity properties of the viscosity solution in order to carry out the verification argument. Under an additional assumption on the penalty function and an additional boundedness condition on the market impact term we prove that the viscosity solution is indeed of class $C^{0,1}$. The proof is based on an asymptotic expansion of the solution to the singular PDE around the terminal time as in [GHS18] and Chapter 3 with the added difficulty that now not only the value functions but also its derivative needs to converge to the market impact term, respectively its derivative when properly rescaled.

The additional regularity of the solution does not only allow us to obtain the optimal trading strategy but also the least favourable martingale measure in feedback form. For small amounts of uncertainty it also allows us to provide a first order approximation of the value function in terms of the solution to the benchmark model without uncertainty. Finally, we prove that our model with factor uncertainty is observationally equivalent to a model without factor uncertainty but increased market risk. This suggests that factor uncertainty increases the rate of liquidation.

2. Multi-dimensional Optimal Trade Execution under Stochastic Resilience

This Chapter is devoted to an analysis of a multi-dimensional portfolio liquidation problem with instantaneous and persistent price impact and stochastic resilience. We establish an existence, uniqueness and approximation of solutions result and analyze the quantitative structure of optimal liquidation strategies in benchmark models with deterministic cost coefficients. Our numerical simulations suggest that the relative sizes of the different impact factors across assets and the correlation between the assets' fundamental values are key determinants of the optimal liquidation strategy. They also suggest that optimal trading rates are typically convex in time with the degree of convexity depending on the instantaneous impact factor. Moreover, optimal strategies are not necessarily of buy-only or sell-only type; the reason is that diversification reduces the portfolio risk. This should be benchmarked against single asset models, where optimal portfolio processes are always monotone if the cost coefficients are deterministic to the best of our knowledge.[1]

This Chapter is structured as follows. The liquidation model is formulated in Section 2.1. The main results are summarized in Section 2.2 where we also provide some numerical simulations. All proofs are carried out in Section 2.3.

2.1. The liquidation model

Throughout we denote by $T \in (0, \infty)$ the liquidation time and fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ that carries a one-dimensional standard Brownian motion $W = (W_t)_{t\in[0,T]}$. We assume that $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by W completed by all the null sets and that $\mathcal{F} = \mathcal{F}_T$.

We consider the problem of a large investor that needs to liquidate a given portfolio $x \in \mathbb{R}^d$ of $d \in \mathbb{N}$ assets with possibly correlated price dynamics within the time horizon $[0, T]$. For $t \in [0, T)$ we denote by $X(t) \in \mathbb{R}^d$ the portfolio that the investor needs to liquidate, and by $\xi(t) \in \mathbb{R}^d$ the rates at which the different stocks are traded at that time. Given a trading strategy ξ, the portfolio position at time

[1] We notice that short sells are not always allowed when closing a client's position. The issue of short sells is discussed in detail in [GS11]. They argue that while short sells are undesirable, they occur only rarely and hence the problem can somehow be ignored, especially since short sell constraints would be difficult to handle mathematically. Our simulations confirm their results: the simulations suggest that short sells occur only rarely and their sizes are rather small if they occur.

$t \in [0, T)$ is given by

$$X(t) = x - \int_0^t \xi(r)\, dr, \quad t \in [0, T].$$

A trading strategy ξ is called *admissible* if it is progressively measurable, belongs to $L^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$ and satisfies the *liquidation constraint*

$$X(T) = 0.$$

It is customary in the liquidation literature to assume that the large investor's transaction price $P(t) \in \mathbb{R}^d$ at time $t \in [0, T]$ can be decomposed into a fundamental asset price $\tilde{P}(t)$ and a market impact term $f(\xi(t))$ as

$$P(t) = \tilde{P}(t) - f\big(\xi(t)\big).$$

We assume that the d-dimensional stochastic process $\tilde{P}$ is a square-integrable Brownian martingale with an essentially bounded covariance matrix Σ. For example, $\Sigma(t) = \sigma(\tilde{P}(t))\sigma(\tilde{P}(t))^{\mathrm{T}}$ for the local stochastic volatility model

$$d\tilde{P}(t) = \sigma\big(\tilde{P}(t)\big) dW(t).$$

The investor aims at minimizing the expected liquidation shortfall plus risk cost. The liquidation shortfall denotes the difference between the book value of the portfolio at the initial time $t = 0$ and the proceeds from trading. Following the majority of the liquidation literature we measure the risk by one-half times the integral of the variance of the portfolio value over the trading period. The risk term penalizes slow liquidation and poorly diversified portfolios. Assuming that the market impact function f can be additively decomposed into an instantaneous and a persistent price impact term as $f(\xi) = \frac{1}{2}\Lambda\xi + Y$, the cost functional is thus given by

$$\begin{aligned} J(x, \xi) &= \text{book value} - \text{expected proceeds from trading} + \text{risk} \\ &= \mathbb{E}\Big[\int_0^T \xi(s) f\big(\xi(s)\big) ds + \frac{1}{2}\int_0^T X(s)^{\mathrm{T}}\Sigma(s)X(s)\, ds\Big] \\ &= \mathbb{E}\Big[\int_0^T \big(\frac{1}{2}\xi(s)^{\mathrm{T}}\Lambda\xi(s) + Y(s)^{\mathrm{T}}\xi(s) + \frac{1}{2}X(s)^{\mathrm{T}}\Sigma(s)X(s)\big)\, ds\Big]. \end{aligned} \tag{2.1}$$

Here, $\Lambda \in \mathcal{S}^d$ is a deterministic positive definite matrix that describes an instantaneous impact factor as in [AC01]. It may be viewed as an additional drift of the benchmark price process resulting from the large investor's trading. Since Λ is not necessarily a diagonal matrix, we allow for spillover effects across different assets; heavily buying/selling a specific asset may well increase/decrease prices of other assets from the same sector. For instance, heavily buying Apple Inc. may increase Microsoft Corporation's price to some extend. The first term in the running cost function in (2.1) describes the cost from instantaneous impact.

The process Y is given by

$$Y(t) = y + \int_0^t \big(-\rho(r)Y(r) + \gamma\xi(r)\big)\,dr, \quad t \in [0,T], \tag{2.2}$$

where $\gamma = \operatorname{diag}(\gamma_i)$ is a positive definite deterministic matrix, and $\rho = \operatorname{diag}(\rho_i)$ is a progressively measurable essentially bounded $\mathcal{S}_+^d$-valued process. The process Y describes the persistent price impacts caused by past trades in block-shaped limit order book markets with constant order book depths $\frac{1}{\gamma_i} > 0$ for the various asset as in [OW13]. The process ρ describes how fast the order books recover from past trades. The fact that γ and ρ are diagonal matrices implies that persistent impacts depend on the trading rates in a particular asset only. This is a reasonable assumption if we interpret Y as an additional spread caused by large investor's trading activities in the respective assets. The second term in the running cost function in (2.1) describes the cost from persistent impact.

2.2. Main results

In this section we state an existence and uniqueness result of solution for the liquidation problem introduced in Section 2.1 and illustrate some of its main quantitative properties. The liquidation problem leads to the following stochastic control problem:

$$\operatorname*{ess\,inf}_{\xi \in L^2_{\mathcal{F}}(t,T;\mathbb{R}^d)} \mathbb{E}\Big[\int_t^T \Big(\frac{1}{2}\xi(s)^{\mathrm{T}}\Lambda\xi(s) + Y(s)^{\mathrm{T}}\xi(s) + \frac{1}{2}X(s)^{\mathrm{T}}\Sigma(s)X(s)\Big)\,ds \Big| \mathcal{F}_t\Big]$$

subject to the state dynamics

$$\begin{cases} X(s) = x - \displaystyle\int_t^s \xi(r)\,dr, \quad s \in [t,T], \\ X(T) = 0, \\ Y(s) = y + \displaystyle\int_t^s \big(-\rho(r)Y(r) + \gamma\xi(r)\big)\,dr, \quad s \in [t,T], \end{cases} \tag{2.3}$$

and the *standing assumption*

$$0 < \Lambda, \gamma = \operatorname{diag}(\gamma_i) \in \mathcal{S}^d; \quad \Sigma, \rho = \operatorname{diag}(\rho_i) \in L^\infty_{\mathcal{F}}(0,T;\mathcal{S}^d_+). \tag{2.4}$$

For any initial state $(t,x,y) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}^d$, the value function of this problem is denoted by

$$\begin{aligned} V(t,x,y) := \operatorname*{ess\,inf}_{\xi \in \mathcal{A}(t,x,y)} \mathbb{E}\Big[\int_t^T \big(\frac{1}{2}\xi(s)^{\mathrm{T}}\Lambda\xi(s) + Y(s)^{\mathrm{T}}\xi(s) \\ + \frac{1}{2}X(s)^{\mathrm{T}}\Sigma(s)X(s)\big)\,ds \Big| \mathcal{F}_t\Big] \end{aligned} \tag{2.5}$$

where the essential infimum is taken over the class $\mathcal{A}(t,x,y)$ of all admissible *liquidation strategies*, that is over all *trading strategies* $\xi \in L^2_{\mathcal{F}}(t,T;\mathbb{R}^d)$ that satisfy the liquidation constraint

$$X(T) = 0.$$

From the dynamics of the persistent impact factor Y we see that

$$\xi(s) = \gamma^{-1}\frac{dY(s)}{ds} + \gamma^{-1}\rho(s)Y(s),$$

where γ^{-1} and ρ are diagonal matrices. Integration by parts yields

$$\begin{aligned}\int_t^T Y(s)^{\mathrm{T}}\xi(s)\,ds = &\int_t^T Y(s)^{\mathrm{T}}\big(\gamma^{-1}dY(s) + \gamma^{-1}\rho(s)Y(s)\,ds\big)\\ =&\frac{1}{2}Y(T)^{\mathrm{T}}\gamma^{-1}Y(T) - \frac{1}{2}y^{\mathrm{T}}\gamma^{-1}y\\ &+\frac{1}{2}\int_t^T Y(s)^{\mathrm{T}}\big(\rho(s)\gamma^{-1} + \gamma^{-1}\rho(s)\big)Y(s)\,ds.\end{aligned}$$

In particular, it is enough to consider the quadratic (instead of linear-quadratic) optimization problem:

$$\begin{aligned}\tilde{V}(t,x,y) := \operatorname*{ess\,inf}_{\xi\in\mathcal{A}(t,x,y)} \mathbb{E}\Bigg[&\frac{1}{2}Y(T)^{\mathrm{T}}\gamma^{-1}Y(T)\\ &+\int_t^T \frac{1}{2}\Big(\xi(s)^{\mathrm{T}}\Lambda\xi(s) + Y(s)^{\mathrm{T}}\big(\rho(s)\gamma^{-1} + \gamma^{-1}\rho(s)\big)Y(s)\\ &\qquad + X(s)^{\mathrm{T}}\Sigma(s)X(s)\Big)\,ds\,\Bigg|\mathcal{F}_t\Bigg]\end{aligned} \tag{2.6}$$

subject to the state dynamics (2.3) and the standing assumption (2.4). Strict convexity of this problem shows that we have at most one solution. The existence of a solution is established in the next subsection.

2.2.1. Existence of solutions

We characterize the value function to the preceding control problem in terms of the unique solution to a matrix-valued BSRDE with singular terminal condition. Our approach is based on an approximation argument. To this end, we consider, for any $n \in \mathbb{N}$, the value function

$$\begin{aligned}\tilde{V}^n(t,x,y) := \operatorname*{ess\,inf}_{\xi\in L^2_{\mathcal{F}}} \mathbb{E}\Bigg[&\frac{n}{2}X(T)^{\mathrm{T}}X(T) + Y(T)^{\mathrm{T}}X(T) + \frac{1}{2}Y(T)^{\mathrm{T}}\gamma^{-1}Y(T)\\ &+\int_t^T \frac{1}{2}\Big(\xi(s)^{\mathrm{T}}\Lambda\xi(s) + Y(s)^{\mathrm{T}}(\rho(s)\gamma^{-1} + \gamma^{-1}\rho(s))Y(s)\\ &\qquad + X(s)^{\mathrm{T}}\Sigma(s)X(s)\Big)\,ds\,\Bigg|\mathcal{F}_t\Bigg]\end{aligned} \tag{2.7}$$

of a corresponding unconstrained optimization problem where the binding liquidation constraint is replaced by a finite penalty of open terminal positions. We solve the unconstrained problem first and then show that the solutions to (2.7) converge to the value function (2.6) as $n \to \infty$.

A pair of random fields $(\tilde{V}^n, \tilde{N}^n) : \Omega \times [0,T) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}$ is called a *classical solution* to (2.7) if it satisfies the following conditions:

- for each $t \in [0,T)$, $\tilde{V}^n(t,x,y)$ is continuously differentiable in x and y,
- for each $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$, $(\tilde{V}^n(t,x,y), \partial_x \tilde{V}^n(t,x,y), \partial_y \tilde{V}^n(t,x,y))_{t\in[0,T]}$ belongs to $L^\infty_{\mathcal{F}}(\Omega; C([0,T]; \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d))$,
- for each $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$, $(\tilde{N}^n(t,x,y))_{t\in[0,T]}$ belongs to $L^2_{\mathcal{F}}(0,T;\mathbb{R})$,
- for all $0 \le t \le s \le T$ and $x, y \in \mathbb{R}^d$ it holds that

$$\begin{cases} \tilde{V}^n(t,x,y) = \tilde{V}^n(s,x,y) - \displaystyle\int_t^s N^n(r,x,y)\,dW(r), \\ \qquad\qquad + \displaystyle\int_t^s \inf_{\xi\in\mathbb{R}^d} \Big\{ \frac{1}{2}\xi^{\mathrm{T}}\Lambda\xi + \frac{1}{2} y^{\mathrm{T}} \big(\rho(r)\gamma^{-1} + \gamma^{-1}\rho(r)\big) y \\ \qquad\qquad\qquad + \frac{1}{2} x^{\mathrm{T}}\Sigma(r) x - \partial_x \tilde{V}^n(r,x,y)^{\mathrm{T}}\xi \\ \qquad\qquad\qquad - \partial_y \tilde{V}^n(r,x,y)^{\mathrm{T}}\big(\rho(r)y - \gamma\xi\big)\Big\}\,dr \\ \tilde{V}^n(T,x,y) = \frac{n}{2} x^{\mathrm{T}} x + y^{\mathrm{T}} x + \frac{1}{2} y^{\mathrm{T}}\gamma^{-1} y. \end{cases} \tag{2.8}$$

The quadratic structure of the control problem suggests the ansatz

$$\begin{aligned} \tilde{V}^n(t,x,y) &= \frac{1}{2}\begin{pmatrix} x^{\mathrm{T}} & y^{\mathrm{T}} \end{pmatrix} Q^n(t) \begin{pmatrix} x \\ y \end{pmatrix} \\ \tilde{N}^n(t,x,y) &= \frac{1}{2}\begin{pmatrix} x^{\mathrm{T}} & y^{\mathrm{T}} \end{pmatrix} M^n(t) \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned} \tag{2.9}$$

for the solution to the HJB equation, where Q^n, M^n are progressively measurable $\mathcal{S}^{2d}$-valued processes. The ansatz reduces our HJB equation (2.8) to the matrix-valued backward stochastic Riccati equation,

$$\begin{aligned} -dQ^n(t) =& \Bigg(- Q^n(t) \begin{pmatrix} -I_d \\ \gamma \end{pmatrix} \Lambda^{-1} \begin{pmatrix} -I_d & \gamma \end{pmatrix} Q^n(t) + Q^n(t) \begin{pmatrix} 0 & 0 \\ 0 & -\rho(t) \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 \\ 0 & -\rho(t) \end{pmatrix} Q^n(t) \quad + \begin{pmatrix} \Sigma(t) & 0 \\ 0 & \gamma^{-1}\rho(t) + \rho(t)\gamma^{-1} \end{pmatrix} \Bigg) dt \\ & - M^n(t)\,dW(t), \\ Q^n(T) =& \begin{pmatrix} nI_d & I_d \\ I_d & \gamma^{-1} \end{pmatrix}. \end{aligned} \tag{2.10}$$

Notice that the terminal value $\begin{pmatrix} nI_d & I_d \\ I_d & \gamma^{-1} \end{pmatrix}$ is nonnegative definite if $n \geq \gamma_{\max}$. In this case, all the coefficients in (2.10) satisfy the requirements in [KT03, Propositions 2.1, 2.2](see also in [Bis76] and [Pen92]). Thus, we have the following theorem.

Theorem 2.2.1. *For every $n \geq \gamma_{\max}$, the BSRDE* (2.10) *has a unique solution*

$$(Q^n, M^n) \in L^\infty_{\mathcal{F}}\left(\Omega; C([0,T]; \mathcal{S}^{2d}_+)\right) \times L^2_{\mathcal{F}}(0,T;\mathcal{S}^{2d}).$$

The value function (2.7) *is of the quadratic form*

$$\tilde{V}^n(t,x,y) = \frac{1}{2}\begin{pmatrix} x^{\mathrm{T}} & y^{\mathrm{T}} \end{pmatrix} Q^n(t) \begin{pmatrix} x \\ y \end{pmatrix}$$

and the optimal $\xi^{n,*}$ is given in feedback form by

$$\xi^{n,*}(t,x,y) = -\Lambda^{-1}\begin{pmatrix} -I_d & \gamma \end{pmatrix} Q^n(t) \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.11}$$

Intuitively, the solution to the (modified) liquidation problem (2.6) should be the limit of the solutions to (2.7) as $n \to \infty$, i.e. be obtained by increasingly penalizing open positions at the terminal time. The following two theorems show that this limit is well-defined and characterizes the value function of our liquidation problem. The proofs are given in Section 2.3 below.

Theorem 2.2.2. *For any $t \in [0,T)$, the limit*

$$Q(t) := \lim_{n \to +\infty} Q^n(t)$$

exists and $\{Q^n(\cdot)\}$ converges compactly to $Q(\cdot)$ on $[0,T)$. Moreover, there exists $M \in L^2_{\mathcal{F}}(0,T^-;\mathcal{S}^{2d})$ such that (Q,M) solves the equation

$$\begin{aligned} -dQ(t) = &\Bigg(- Q(t)\begin{pmatrix} -I_d \\ \gamma \end{pmatrix}\Lambda^{-1}\begin{pmatrix} -I_d & \gamma \end{pmatrix} Q(t) + Q(t)\begin{pmatrix} 0 & 0 \\ 0 & -\rho(t) \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ 0 & -\rho(t) \end{pmatrix} Q(t) + \begin{pmatrix} \Sigma(t) & 0 \\ 0 & \gamma^{-1}\rho(t) + \rho(t)\gamma^{-1} \end{pmatrix}\Bigg) dt \\ &- M(t)\, dW(t). \end{aligned} \tag{2.12}$$

on $[0,T)$. Furthermore,

$$\liminf_{t \to T} |Q(t)| = +\infty.$$

By Theorem 2.2.2 we also obtain the existence of the limit of the optimal strategies as $n \to \infty$:

$$\xi^*(t,x,y) := \lim_{n \to \infty} \xi^{n,*}(t,x,y) = -\Lambda^{-1}\begin{pmatrix} -I_d & \gamma \end{pmatrix} Q(t) \begin{pmatrix} x \\ y \end{pmatrix}.$$

Thus, the optimal trading strategy is given in terms of a linear combination of the positions in the various assets as well as the spreads in the markets for the different assets. We will see that ξ^* is usually not of buy-only or sell-only type. The following is the main result of this paper.

Theorem 2.2.3. *Let Q be the limit given in Theorem 2.2.2. Then the value function* (2.6) *is given by*

$$\tilde{V}(t,x,y) = \frac{1}{2}\begin{pmatrix} x^{\mathrm{T}} & y^{\mathrm{T}} \end{pmatrix} Q(t) \begin{pmatrix} x \\ y \end{pmatrix}$$

and the optimal control in feedback form is given by

$$\xi^*(t,x,y) = -\Lambda^{-1}\begin{pmatrix} -I_d & \gamma \end{pmatrix} Q(t) \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.13}$$

Corollary 2.2.4. *Let Q be the limit given in Theorem 2.2.2. Then the value function* (2.5) *is given by*

$$V(t,x,y) = \frac{1}{2}\begin{pmatrix} x^{\mathrm{T}} & y^{\mathrm{T}} \end{pmatrix} Q(t) \begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{2} y^{\mathrm{T}}\gamma^{-1} y \tag{2.14}$$

and the optimal control in feedback form is given by (2.13)*.*

2.2.2. Numerical analysis

It is difficult to obtain analytic results on the dependence of the optimal liquidation strategy on the model parameters. For instance, the optimal strategy depends both directly on the market depth parameter γ as well as indirectly through the dependence of the solution of the Riccati BSDE on γ. In order to get some insight into the nature of the optimal liquidation strategy we report in this section some simulation results for a benchmark model with two assets and deterministic cost coefficients. To simplify the exposition, we assume that there is no cross asset price impact and hence assume that Λ is a diagonal matrix. This is a reasonable assumption if we trading stocks from different sections such as Apple Incorporation and Ford Motor Company; it might not be a reasonable assumption if we are trading Apple Incorporation and Microsoft Corporation. We allow for correlated (e.g. on macroeconomic factors) fundamental prices, though. We then choose

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \sigma_1^2 & k\sigma_1\sigma_2 \\ k\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

If all the cost coefficients are deterministic constants, the stochastic Riccati equations reduce to a multi-dimensional ODE system that can be solved numerically using the MATLAB package `bvpsuite` [KKP$^+$10].

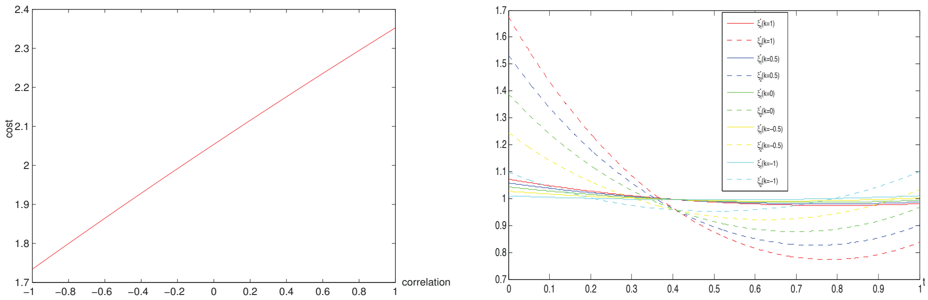

Figure 2.1.: Dependence of the cost (left) and the optimal trading strategies (right) on the correlation k for the parameters $T = 1$, $x_1 = x_2 = 1, y_1 = y_2 = 0$, $\lambda_1 = 10, \lambda_2 = 1$, $\sigma_1 = \sigma_2 = \gamma_1 = \gamma_2 = \rho_1 = \rho_2 = 1$.

Figure 2.1[2] shows that the cost increases in the correlation of the assets' fundamental price processes. This is natural as a negative correlation reduces risk costs. We also see that for our choice of model parameters the more liquid asset is liquidated at a much faster rate than the less liquid one and that the initial liquidation rate increases in the correlation. Both results are intuitive; fast liquidation reduces risk cost and the cost savings are increasing in the correlation. Moreover, the less liquid asset is liquidated at an almost constant rate while the more liquid asset is liquidated at a convex rate with the degree of convexity decreasing in the correlation.

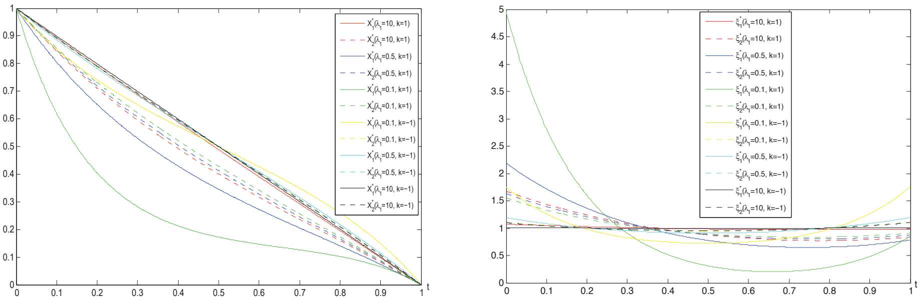

Figure 2.2.: Dependence of the optimal positions (left) and the trading strategies (right) on the instantaneous impact factor λ_1 for the parameters $T = 1$, $x_1 = x_2 = 1$, $y_1 = y_2 = 0$, $\lambda_2 = 1, \sigma_1 = \sigma_2 = \gamma_1 = \gamma_2 = \rho_1 = \rho_2 = 1$.

The convexity of the optimal liquidation strategy is consistent with the single asset case analyzed in [GH17]. There, it is shown that when the instantaneous market impact is small, the optimal liquidation strategy resembles a strategy with block trades: the (single) asset is liquidated at a very high rate initially (to benefit

[2]Since the characters in Figures 2.1-2.5 are too small to recognize, we summarize the legend labels in Appendix A.1 for the reader's convenience

from resilience) and close to the terminal time (where the cost of a widening of the spread is low). Similar results for the 2-dimensional case are shown in Figure 2.2 where the dependence of the value function (left) and the optimal liquidation strategy (right) on the impact factors λ_1 is depicted.

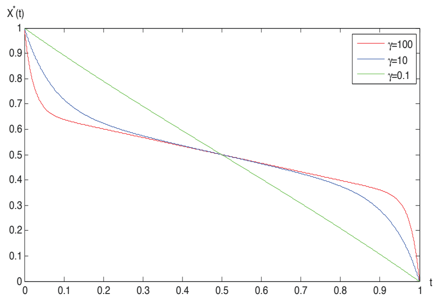

Figure 2.3.: Dependence of the optimal position (left) and the trading strategy (right) in a single asset model on the persistent impact factor γ for the parameters $T = 1,\ x = 1, y = 0,\ \lambda = 0.1, \sigma = 0, \rho = 1$.

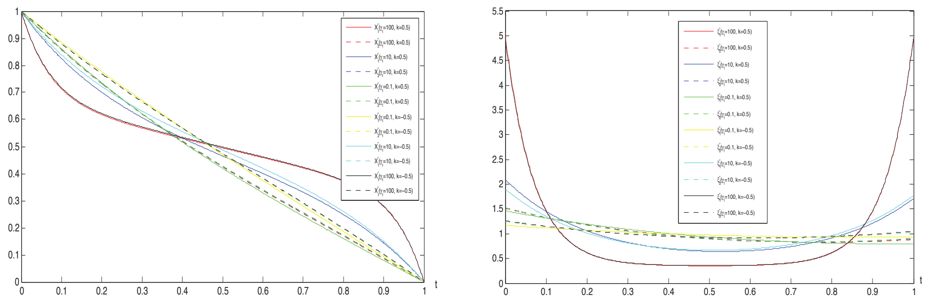

Figure 2.4.: Dependence of the optimal positions (left) and the trading strategies (right) on the persistent impact factor γ_1 for the parameters $T = 1$, $x_1 = x_2 = 1,\ y_1 = y_2 = 0, \lambda_1 = \lambda_2 = 1,\ \sigma_1 = \sigma_2 = \gamma_2 = \rho_1 = \rho_2 = 1$.

While the initial trading rate *decreases* if the instantaneous impact factor increases, our simulations suggest that it *increases* with the persistent impact factors. This effect can already been seen in the single asset case as shown in Figure 2.3. Simulations for the 2-dimensional case are shown in Figure 2.4. If the persistent impact factor is large, early trading benefits from resilience. In fact, if there is no resilience and if the persistent impact dominates the cost function to the extend that we may drop the instantaneous impact and risk cost, the resulting Lagrange equation is zero and any liquidation strategy is optimal.

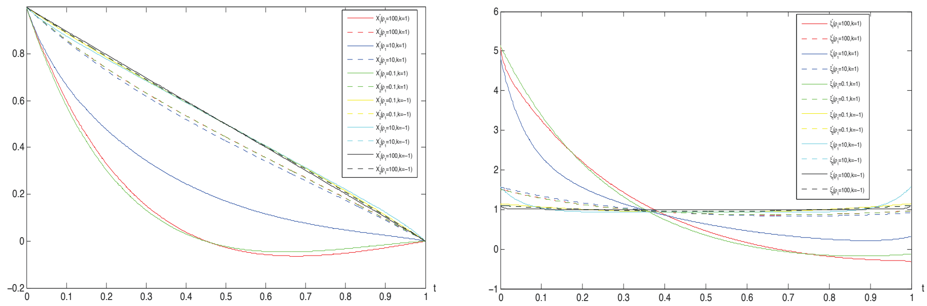

Figure 2.5.: Dependence of the optimal positions (left) and the trading strategies (right) on the resilience factor ρ_1 for the parameters $T = 1, x_1 = x_2 = 1$, $y_1 = y_2 = 0, \lambda_1 = 0.1$, $\lambda_2 = \sigma_1 = \sigma_2 = \rho_2 = \gamma_1 = \gamma_2 = 1$.

The dependence of the optimal solution on the resilience factor ρ_1 is shown in Figure 2.5. Although we observe again that the optimal strategy is convex, the dependence of the convexity on the strength of resilience is less clear than that on the other impact factors. We also see from that figure (red and green curves) that short positions can not be excluded; if the assets are strongly positively correlated, a negative position in one asset may well be beneficial in order to balance the portfolio risk, i.e. to minimize the risk term $X(s)^{\mathrm{T}}\Sigma(s)X(s)$.

2.3. Proofs

In this section, we give the proofs of Theorems 2.2.2 and 2.2.3. In a first step, we bound (with respect to the partial order on the cone of nonnegative definite matrices) the processes Q^n by matrix-valued processes whose limiting behaviour at the terminal time can be inferred from a one-dimensional benchmark model (Lemma 2.3.1). This will enable us to prove the existence of the limit $\lim_{n\to\infty} Q^n$ (Theorem 2.2.2). In a second step, we establish upper and lower bounds for

$$\sqrt{\Lambda^{-1}} \begin{pmatrix} -I_d & \gamma \end{pmatrix} Q^n \begin{pmatrix} -I_d \\ \gamma \end{pmatrix} \sqrt{\Lambda^{-1}}$$

near the terminal time (Proposition 2.3.4), from which we will infer the convergence of the strategies $\{\xi^{n,*}\}$ to an admissible liquidation strategy.

Notation. The following notion will be useful. For a generic matrix $Q \in S^{2d}$, we write

$$Q_{2d\times 2d} = \begin{pmatrix} A_{d\times d} & B_{d\times d} \\ B^{\mathrm{T}}_{d\times d} & C_{d\times d} \end{pmatrix} \tag{2.15}$$

and define D, E, F as follows:

$$D := \begin{pmatrix} -I_d & \gamma \end{pmatrix} Q \begin{pmatrix} -I_d \\ 0 \end{pmatrix} = (A - \gamma B^{\mathrm{T}}), \quad E := \begin{pmatrix} -I_d & \gamma \end{pmatrix} Q \begin{pmatrix} 0 \\ I_d \end{pmatrix} = (\gamma C - B)$$

and

$$F := \begin{pmatrix} -I_d & \gamma \end{pmatrix} Q \begin{pmatrix} -I_d \\ \gamma \end{pmatrix} = D + E\gamma.$$

2.3.1. A priori estimates

If $d = 1$, A, B, C are one-dimensional, then $Q = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$ and the system (2.10) simplifies to the three-dimensional BSRDEs:

$$\begin{cases} -dA(t) = \Big(\sigma(t) - \lambda^{-1}\big(A(t) - \gamma B(t)\big)^2\Big)dt - M^A(t)\,dW(t) \\ -dB(t) = \Big(-\rho(t)B(t) + \lambda^{-1}\big(\gamma C(t) - B(t)\big)\big(A(t) - \gamma B(t)\big)\Big)dt \\ \qquad\qquad -M^B(t)\,dW(t) \\ -dC(t) = \Big(-2\rho(t)C(t) + 2\rho(t)\gamma^{-1} - \lambda^{-1}\big(\gamma C(t) - B(t)\big)^2\Big)dt \\ \qquad\qquad -M^C(t)\,dW(t) \\ \quad A(T) = n, \quad B(T) = 1, \quad C(T) = \gamma^{-1}. \end{cases} \tag{2.16}$$

Analogous to the a priori estimates in Lemma 3.1 and Proposition 3.2 of [GH17], we have the following bounds on $[0, T]$:

$$\begin{aligned} \frac{\gamma}{e^{\lambda^{-1}\gamma(T-t)}(1 + \frac{\gamma}{n-\gamma}) - 1} &\le D(t) \le \lambda\kappa \coth\left(\kappa(T-t) + \operatorname{arccoth}\frac{\lambda^{-1}(n-\gamma)}{\kappa}\right), \\ e^{-\|\rho\|_{L^\infty}(T-t)} &\le B(t) \le 1, \\ 0 &\le E(t) \le 1, \\ 0 &\le C(t) \le \gamma^{-1}, \end{aligned}$$

where $\kappa := \sqrt{2\lambda^{-1}\max\{\|\sigma\|_{L^\infty}, \gamma\|\rho\|_{L^\infty}\}}$. Thus,

$$\begin{cases} A(t), F(t) \ge \dfrac{\gamma}{e^{\lambda^{-1}\gamma(T-t)}(1 + \frac{\gamma}{n-\gamma}) - 1}, \\ A(t), F(t) \le \lambda\kappa \coth\left(\kappa(T-t) + \operatorname{arccoth}\dfrac{\lambda^{-1}(n-\gamma)}{\kappa}\right) + \gamma, \\ \gamma^{-1}e^{-\|\rho\|_{L^\infty}(T-t)} \le C(t) \le \gamma^{-1}. \end{cases} \tag{2.17}$$

A first (rough) estimate

Let $\tilde{V}^n(t, x, y)$ be defined as in (2.7) and let $\tilde{V}^{n,max}(t, x, y), \tilde{V}^{n,min}(t, x, y)$ be also defined as in (2.7) but with (Λ, Σ) replaced by $(\lambda_{\max} I_d, |\Sigma(t)| I_d)$ and $(\lambda_{\min} I_d, 0)$, respectively. Then it follows from Theorem 2.2.1 that

$$\tilde{V}^n(t, x, y) = \begin{pmatrix} x^{\mathrm{T}} & y^{\mathrm{T}} \end{pmatrix} Q^n(t) \begin{pmatrix} x \\ y \end{pmatrix}$$

$$
\begin{aligned}
\tilde{V}^{n,max}(t,x,y) &= \begin{pmatrix} x^{\mathrm{T}} & y^{\mathrm{T}} \end{pmatrix} Q^{n,max}(t) \begin{pmatrix} x \\ y \end{pmatrix} \\
\tilde{V}^{n,min}(t,x,y) &= \begin{pmatrix} x^{\mathrm{T}} & y^{\mathrm{T}} \end{pmatrix} Q^{n,min}(t) \begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}
$$

The quadratic cost function in (2.7) implies

$$
\tilde{V}^{n,\min}(t,x,y) \leq \tilde{V}^{n}(t,x,y) \leq \tilde{V}^{n,\max}(t,x,y),
$$

and hence

$$
Q^n_{\min}(t) \leq Q^n(t) \leq Q^n_{\max}(t) \quad \text{for all } t \in [0,T]. \tag{2.18}
$$

Similarly, we obtain that the processes Q^n, $Q^{n,\max}$ and $Q^{n,\min}$ are nondecreasing in n, due to the monotonicity of the terminal condition.

If Λ, Σ were diagonal matrices, the control problem (2.7) would separate into d subsystems, whose BSRDE system is similar to the solution of three-dimensional system (2.16). So the matrices $Q^n_{\max}, Q^n_{\min}$ are of the form

$$
\begin{pmatrix}
A_1 & & & B_1 & & \\
& \ddots & & & \ddots & \\
& & A_d & & & B_d \\
B_1 & & & C_1 & & \\
& \ddots & & & \ddots & \\
& & B_d & & & C_d
\end{pmatrix},
$$

where each triple (A_i, B_i, C_i) solves the BSRDE (2.16) if $(\lambda, \gamma, \sigma, \rho)$ is replaced by $(\lambda_{\max}, \gamma_i, |\Sigma|, \rho_i)$ and $(\lambda_{\min}, \gamma_i, 0, \rho_i)$, respectively. Combining the inequality (2.18) with the a priori estimates (2.17) and recalling that A^n, B^n, C^n, F^n come from the decomposition of the matrix Q_n, we obtain the following result.

Lemma 2.3.1. *For every $n \geq \gamma_{max}$, the following a priori estimates hold for all $t \in [0,T]$:*

$$
\begin{aligned}
&\operatorname{diag}(\underline{A}^n_i) \leq A^n_{\min} \leq A^n \leq A^n_{\max} \leq \operatorname{diag}(\overline{A}_i), \\
&\operatorname{diag}(\underline{C}^n_i) \leq C^n_{\min} \leq C^n \leq C^n_{\max} \leq \gamma^{-1}, \\
&\operatorname{diag}(\underline{F}^n_i) \leq F^n_{\min} \leq F^n \leq F^n_{\max} \leq \operatorname{diag}(\overline{F}_i).
\end{aligned} \tag{2.19}
$$

and $B^n_{\max} \leq I_d$ where

$$
\begin{aligned}
\underline{C}^n_i &= \gamma_i^{-1} e^{-\|\rho_i\|_{L^\infty}(T-t)}, \\
\underline{A}^n_i &= \underline{F}^n_i = \frac{\gamma_i}{e^{\lambda_{\min}^{-1}\gamma_i(T-t)}(1 + \frac{\gamma_i}{n-\gamma_i}) - 1}, \\
\overline{A}_i &= \overline{F}_i = \lambda_{\max}\kappa_i \coth\big(\kappa_i(T-t)\big) + \gamma_i, \\
\kappa_i &= \sqrt{2\lambda_{\max}^{-1}\max\{\|\Sigma\|_{L^\infty}, \gamma_i\|\rho_i\|_{L^\infty}\}}.
\end{aligned}
$$

A second (finer) estimate

We now extend arguments given in [Kra11, Section 2.2.4] to bound the processes:

$$\sqrt{\Lambda^{-1}}\begin{pmatrix}-I_d & \gamma\end{pmatrix} Q^n \begin{pmatrix}-I_d \\ \gamma\end{pmatrix} \sqrt{\Lambda^{-1}} = \sqrt{\Lambda^{-1}} F^n \sqrt{\Lambda^{-1}}.$$

Multiplying $\begin{pmatrix}-I_d & \gamma\end{pmatrix}$ on the left and $\begin{pmatrix}-I_d \\ \gamma\end{pmatrix}$ on the right in (2.10), we see that F^n satisfies

$$\begin{aligned}
-dF^n(t) =& \Bigg(\begin{pmatrix}-I_d & \gamma\end{pmatrix} \left(Q^n(t) \begin{pmatrix}0 & 0 \\ 0 & -\rho(t)\end{pmatrix} + \begin{pmatrix}0 & 0 \\ 0 & -\rho(t)\end{pmatrix} Q^n(t) \right) \begin{pmatrix}-I_d \\ \gamma\end{pmatrix} \\
& - F^n(t)\Lambda^{-1}F^n(t) + \Sigma(t) + 2\gamma\rho(t) \Bigg) dt \\
& - \begin{pmatrix}-I_d & \gamma\end{pmatrix} M^n(t) \begin{pmatrix}-I_d \\ \gamma\end{pmatrix} dW(t), \\
F^n(T) =& nI_d - \gamma.
\end{aligned} \tag{2.20}$$

Our goal is to bound the processes F^n by the solutions to deterministic RDEs. To this end, we first prove that the process

$$\begin{pmatrix}-I_d & \gamma\end{pmatrix} \left(Q^n \begin{pmatrix}0 & 0 \\ 0 & -\rho\end{pmatrix} + \begin{pmatrix}0 & 0 \\ 0 & -\rho\end{pmatrix} Q^n \right) \begin{pmatrix}-I_d \\ \gamma\end{pmatrix}$$

can be bounded from below and above by $-2F^n$ and $2F^n$, respectively.

Lemma 2.3.2. *Set*

$$\begin{aligned}
n_0 :&= \max\{\lambda_{\min}(\sqrt{1+\alpha}+1) + \gamma_{\min}, (\beta+1)\gamma_{\max} + 1\}, \\
\beta :&= 3 + 2\|\rho\|_{L^\infty}^2, \\
\alpha :&= \frac{\|\Sigma\|_{L^\infty} + 2\gamma_{max}\|\rho\|_{L^\infty}}{\lambda_{\min}},
\end{aligned} \tag{2.21}$$

and

$$T_0 := \max_i \left\{ T - \frac{\lambda_{\min}}{\gamma_i(\frac{1}{2}+\beta)} \frac{n_0 - (\beta+1)\gamma_i}{n_0 - \frac{\gamma_i}{2}} \right\} \vee 0. \tag{2.22}$$

For our choice of n_0, we have $T_0 < T$. Then, for any $n \geq n_0, t \in [T_0, T]$,

$$-2F^n \leq \begin{pmatrix}-I_d & \gamma\end{pmatrix} \left(Q^n \begin{pmatrix}0 & 0 \\ 0 & -\rho\end{pmatrix} + \begin{pmatrix}0 & 0 \\ 0 & -\rho\end{pmatrix} Q^n \right) \begin{pmatrix}-I_d \\ \gamma\end{pmatrix} \leq 2F^n. \tag{2.23}$$

Proof. Using the matrix decomposition introduced prior to Section 2.3.1, we need to prove that

$$-2F^n \leq -\big(-\gamma\rho(B^n)^{\mathrm{T}} - B^n\rho\gamma + \gamma C^n\gamma\rho + \rho\gamma C^n\gamma \big) \leq 2F^n, \quad t \in [T_0, T].$$

Since Q^n is nonnegative definite,

$$\begin{aligned}&A^n - (2I_d - \rho)\gamma(B^n)^{\mathrm{T}} - B^n\gamma(2I_d - \rho) + (2I_d - \rho)\gamma C^n\gamma(2I_d - \rho)\\ =&\begin{pmatrix}-I_d & (2I_d - \rho)\gamma\end{pmatrix} Q^n(t) \begin{pmatrix}-I_d \\ \gamma(2I_d - \rho)\end{pmatrix} \geq 0.\end{aligned}$$

In view of (2.15) it follows that,

$$\begin{aligned}&2F^n + \gamma\rho(B^n)^{\mathrm{T}} + B^n\rho\gamma - \gamma C^n\gamma\rho - \rho\gamma C^n\gamma\\ =&\big(A^n - (2I_d - \rho)\gamma(B^n)^{\mathrm{T}} - B^n\gamma(2I_d - \rho) + (2I_d - \rho)\gamma C^n\gamma(2I_d - \rho)\big)\\ &+ A^n - (2I_d - \rho)\gamma C^n\gamma(2I_d - \rho) + 2\gamma C^n\gamma - \gamma C^n\gamma\rho - \rho\gamma C^n\gamma\\ \geq& A^n + (I_d + \rho)\gamma C^n\gamma(I_d + \rho) - 2\rho\gamma C^n\gamma\rho - 3\gamma C^n\gamma\\ \geq& A^n - 2\rho\gamma C^n\gamma\rho - 3\gamma C^n\gamma. \qquad (2.24)\end{aligned}$$

For $n > \gamma_{\max}$,

$$\underline{A}_i^n = \frac{\gamma_i}{e^{\lambda_{\min}^{-1}\gamma_i(T-t)}(1 + \frac{\gamma_i}{n-\gamma_i}) - 1} \geq \frac{\lambda_{\min}}{T - t + \frac{\lambda_{\min}}{n-\gamma_i/2}} - \frac{\gamma_i}{2}.$$

In fact, $\frac{\underline{A}_i^n}{\lambda_{\min}}$ can be expressed as the solution of

$$\begin{cases} y' = y^2 + \dfrac{\gamma_i}{\lambda_{\min}} y, \\ y(T) = \dfrac{n - \gamma_i}{\lambda_{\min}}. \end{cases}$$

Then from [Kra11, Corollary 2.2.3], we have that

$$\frac{\underline{A}_i^n}{\lambda_{\min}} \geq \frac{1}{T - t + \frac{\lambda_{\min}}{n-\gamma_i/2}} - \frac{\gamma_i}{2\lambda_{\min}}.$$

Set

$$f(t,n) = \frac{\lambda_{\min}}{T - t + \frac{\lambda_{\min}}{n-\gamma_i/2}} - \frac{\gamma_i}{2} - \beta\gamma_i.$$

It is easy to check that

$$f(T_0, n_0) \geq \frac{\lambda_{\min}}{\frac{\lambda_{\min}}{\gamma_i(\frac{1}{2}+\beta)} \frac{n_0-(\beta+1)\gamma_i}{n_0-\frac{\gamma_i}{2}} + \frac{\lambda_{\min}}{n_0-\gamma_i/2}} - \frac{\gamma_i}{2} - \beta\gamma_i = 0.$$

Since f is increasing in t and n, we have $f(t,n) \geq 0$ for $t \in [T_0, T], n \geq n_0$. Moreover, by Lemma 2.3.1, we know that

$$\gamma C^n\gamma \leq \gamma \quad \textit{and} \quad A^n \geq \mathrm{diag}(\underline{A}_i^n).$$

Therefore,

$$A^n - 2\rho\gamma C^n\gamma\rho - 3\gamma C^n\gamma \geq \mathrm{diag}(\underline{A}_i^n) - \beta\gamma \geq 0. \qquad (2.25)$$

This yields the left inequality in (2.23). For the right inequality, notice that similarly to (2.24),

$$\begin{aligned} &2F^n - \gamma\rho(B^n)^{\mathrm{T}} - B^n\rho\gamma + \gamma C^n\gamma\rho + \rho\gamma C^n\gamma \\ \geq &A^n + (I_d - \rho)\gamma C^n\gamma(I_d - \rho) - 2\rho\gamma C^n\gamma\rho - 3\gamma C^n\gamma. \end{aligned}$$

Hence, the right inequality also follows from (2.25). □

From [Kra11, Section 2.2.2], we have the following lemma.

Lemma 2.3.3. *Let $n > n_1$, where*

$$n_1 := \max\{\lambda_{\min}(\sqrt{1+\alpha}+1) + \gamma_{\min}, \gamma_{\max}\}$$

and let T_0 be as in equation (2.22). Then the terminal value problems

$$-dK(t) = -\{K(t)^2 - 2K(t) - \alpha I_d\}\,dt, \quad K(T) = \frac{n-\gamma_{\min}}{\lambda_{\min}} I_d \tag{2.26}$$

and

$$-dK(t) = -\{K(t)^2 + 2K(t)\}\,dt, \quad K(T) = \frac{n-\gamma_{\max}}{\lambda_{\max}} I_d \tag{2.27}$$

with

$$\alpha = \frac{\|\Sigma\|_{L^\infty} + 2\gamma_{\max}\|\rho\|_{L^\infty}}{\lambda_{\min}} \tag{2.28}$$

possess unique solutions $K^n_{\max}$, respectively $K^n_{\min}$, on $[T_0, T]$. They are given by

$$\begin{aligned} K^n_{\max}(t) &= p^n(t) I_d, \\ K^n_{\min}(t) &= q^n(t) I_d, \end{aligned}$$

where

$$\begin{aligned} p^n(t) &= \sqrt{1+\alpha}\coth(\sqrt{1+\alpha}(T-t) + \kappa^n_1) + 1, \\ q^n(t) &= \coth(T - t + \kappa^n_2) - 1, \end{aligned} \tag{2.29}$$

with

$$\begin{aligned} \kappa^n_1 &= \operatorname{arcoth}\Big(\frac{\frac{n-\gamma_{\min}}{\lambda_{\min}} - 1}{\sqrt{1+\alpha}}\Big), \\ \kappa^n_2 &= \operatorname{arcoth}\Big(\frac{n-\gamma_{\max}}{\lambda_{\max}} + 1\Big). \end{aligned}$$

The matrices $K^n_{\max}, K^n_{\min}$ in Lemma 2.3.3 turn out to be the desired bounds for $\sqrt{\Lambda^{-1}}F^n\sqrt{\Lambda^{-1}}$ near the terminal time.

Proposition 2.3.4. *Let n_0 be as in (2.21) and T_0 be as in (2.22). Then for $n > n_0$,*

$$q^n(t) I_d \leq \sqrt{\Lambda^{-1}}F^n\sqrt{\Lambda^{-1}} \leq p^n(t) I_d, \quad t \in [T_0, T]. \tag{2.30}$$

Proof. Let
$$\hat{F}^n = \sqrt{\Lambda^{-1}} F^n \sqrt{\Lambda^{-1}}.$$
Multiplying $\sqrt{\Lambda^{-1}}$ both on the left and right of (2.20), we see that $\hat{F}^n$ solves

$$\begin{aligned}-d\hat{F}^n(t) = &\Big(G(t) - \hat{F}^n(t)\cdot\hat{F}^n(t) + \sqrt{\Lambda^{-1}}(\Sigma(t)+2\gamma\rho)\sqrt{\Lambda^{-1}}\Big)\,dt \\ &- \sqrt{\Lambda^{-1}}\begin{pmatrix}-I_d & \gamma\end{pmatrix} M^n(t)\begin{pmatrix}-I_d \\ \gamma\end{pmatrix}\sqrt{\Lambda^{-1}}\,dW(t),\\ \hat{F}^n(T) =&\sqrt{\Lambda^{-1}}(nI_d-\gamma)\sqrt{\Lambda^{-1}},\end{aligned}$$

where

$$G := \sqrt{\Lambda^{-1}}\begin{pmatrix}-I_d & \gamma\end{pmatrix}\left(Q^n\begin{pmatrix}0 & 0\\ 0 & -\rho\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & -\rho\end{pmatrix}Q^n\right)\begin{pmatrix}-I_d \\ \gamma\end{pmatrix}\sqrt{\Lambda^{-1}}.$$

From Lemma 2.3.2, we know that on $[T_0, T]$,

$$-2\hat{F}^n \le G(t) \le 2\hat{F}^n.$$

In terms of α given in (2.28),

$$\begin{aligned}&0 \le \sqrt{\Lambda^{-1}}(\Sigma(t)+2\gamma\rho)\sqrt{\Lambda^{-1}} \le \alpha I_d,\\ &\frac{n-\gamma_{\max}}{\lambda_{\max}} I_d \le \sqrt{\Lambda^{-1}}(nI_d-\gamma)\sqrt{\Lambda^{-1}} \le \frac{n-\gamma_{\min}}{\lambda_{\min}} I_d.\end{aligned}$$

Using similar argument to Section 2.3.1, we obtain a comparison principle to the solution of BSRDEs by the verification argument stated in [KT03, Proposition 2.2]. Hence, we have
$$K^n_{\min}(t) \le \hat{F}^n(t) \le K^n_{\max}(t), \quad t\in[T_0,T]$$
where $K^n_{\max}, K^n_{\min}$ are the solutions to equations (2.26), (2.27). Hence the assertion follows from the fact that $K^n_{\max} = p^n I_d$, $K^n_{\min}(t) = q^n I_d$. □

The preceding proposition established upper and lower bounds for the processes $\sqrt{\Lambda^{-1}}F^n\sqrt{\Lambda^{-1}}$ in terms of the functions q^n and p^n on $[T_0, T]$. For analytical convenience we extend these functions and the bounds to the whole interval $[0, T]$ by putting

$$\begin{aligned}q^n(t) &:= \lambda_{\max}^{-1}\min_i\{\underline{F}_i^{n_0}(0)\},\\ p^n(t) &:= \lambda_{\min}^{-1}\max_i\{\overline{F}_i(T_0)\},\end{aligned} \tag{2.31}$$

for $t\in[0,T_0)$ and $n > n_0$. It's worthy to note that all quantities are nonnonegative in the equations above. Then from (2.19) we have

$$q^n(t)I_d \le \sqrt{\Lambda^{-1}}F^n(t)\sqrt{\Lambda^{-1}} \le p^n(t)I_d, \quad t\in[0,T].$$

2.3.2. The unconstrained problems

In this section, we analyze the unconstrained optimization problem. The verification result for the unconstrained problem can be taken from [Bis76, Pen92, KT03]. However, we still need to establish suitable a priori estimates for Q_n and $\xi^{n,*}$ in order to establish the convergence of the solutions to the unconstrained problems (2.6) to the solution to the original liquidation problem.

For any initial state $(t,x,y) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}^d$, the dynamics of the state process $(X^{n,*}, Y^{n,*})$ under the candidate strategy $\xi^{n,*}$ is given by:

$$\begin{cases} dX^{n,*}(s) = \big(-\Lambda^{-1} D^n(s) X^{n,*}(s) + \Lambda^{-1} E^n(s) Y^{n,*}(s) \big)\, ds, \\ dY^{n,*}(s) = \Big(-\big(\rho(s) + \gamma \Lambda^{-1} E^n(s)\big) Y^{n,*}(s) + \gamma \Lambda^{-1} D^n(s) X^{n,*}(s) \Big)\, ds. \end{cases}$$

In particular, $dY^{n,*}(s) = -\gamma\, dX^{n,*}(s) - \rho(s) Y^{n,*}(s)\, ds$, and hence

$$\begin{aligned} Y^{n,*}(s) = & -\gamma X^{n,*}(s) + e^{-\int_t^s \rho(r)dr}(y + \gamma x) \\ & + \int_t^s e^{-\int_u^s \rho(r)dr} \gamma \rho(u) X^{n,*}(u)\, du, \end{aligned} \tag{2.32}$$

where $e^{-\int_t^s \rho(r)dr} = \operatorname{diag}(e^{-\int_t^s \rho_i(r)dr})$. Thus,

$$\begin{aligned} dX^{n,*}(s) = & \Big(-\Lambda^{-1}\big(D^n(s) + E^n(s)\gamma\big) X^{n,*}(s) + \Lambda^{-1} E^n(s) e^{-\int_t^s \rho(r)dr}(y + \gamma x) \\ & + \Lambda^{-1} E^n(s) \int_t^s e^{-\int_u^s \rho(r)dr} \gamma \rho(u) X^{n,*}(u)\, du \Big) ds. \end{aligned} \tag{2.33}$$

In order to solve this equation, we introduce the d-dimensional fundamental matrix $\Phi^n(t,s)$. It is given by the unique solution of the ODE system

$$\begin{cases} d\Phi^n(t,s) = -\Lambda^{-1}\big(D^n(s) + E^n(s)\gamma\big)\Phi^n(t,s)\, ds, \\ \Phi^n(t,t) = I_d. \end{cases} \tag{2.34}$$

The inverse $(\Phi^n)^{-1}$ exists and satisfies

$$\begin{cases} d\Phi^n(t,s)^{-1} = \Phi^n(t,s)^{-1}\Lambda^{-1}\big(D^n(s) + E^n(s)\gamma\big)\, ds, \\ \Phi^n(t,t)^{-1} = I_d. \end{cases}$$

The following lemma establishes norm bounds on the fundamental solution and its inverse.

In what follows, for better readability, we shorten the notations $\operatorname{ess\,sup}_{\omega \in \Omega} \sup_{n > n_0}$ to $\sup_n$ and $\operatorname{ess\,sup}_{\omega \in \Omega} \sup_{n > n_0, s \in [t,T]}$ to $\sup_{n,s}$.

Lemma 2.3.5. *Let us fix $t \in [0,T)$. For all $t \le s \le T$,*

$$|\Phi^n(t,s)|^2 \le d \frac{\lambda_{\max}}{\lambda_{\min}} \exp\left(-2 \int_t^s q^n(u)\, du \right),$$

$$|\Phi^n(t,s)^{-1}|^2 \leq d\frac{\lambda_{\max}}{\lambda_{\min}} \exp\left(2\int_t^s p^n(u)\,du\right). \tag{2.35}$$

In particular,

$$\sup_{n,s} |\Phi^n(t,s)| < +\infty, \tag{2.36}$$

due to Lemma A.2.1.

Proof. Let $\Phi^n(t,s) = (\phi_1^n(t,s), \phi_2^n(t,s), \cdots, \phi_d^n(t,s))^{\mathrm{T}}$. For $i = 1, ..., d$, we obtain by Proposition 2.3.4 and (2.31) that

$$\begin{aligned}
\frac{d\phi_i^n(t,s)^{\mathrm{T}}\Lambda\phi_i^n(t,s)}{ds} &= -2\phi_i^n(t,s)^{\mathrm{T}}(D^n(s) + E^n(s)\gamma)\phi_i^n(t,s) \\
&= -2\phi_i^n(t,s)^{\mathrm{T}}\sqrt{\Lambda}\hat{F}^n(s)\sqrt{\Lambda}\phi_i^n(t,s) \\
&\leq -2q^n(s)\phi_i^n(t,s)^{\mathrm{T}}\Lambda\phi_i^n(t,s).
\end{aligned}$$

Since $q^n(s)$ is discontinuous at T_0, if $t < T_0 < s$ we divide the interval $[t,s)$ into two subintervals $[t,T_0), [T_0,s)$. On each subinterval the assumptions of Gronwall's inequality are satisfied. Hence,

$$\begin{aligned}
\phi_i^n(t,T_0)^{\mathrm{T}}\Lambda\phi_i^n(t,T_0) &\leq \phi_i^n(t,t)^{\mathrm{T}}\Lambda\phi_i^n(t,t)\exp\left(-2\int_t^{T_0} q^n(u)\,du\right), \\
\phi_i^n(t,s)^{\mathrm{T}}\Lambda\phi_i^n(t,s) &\leq \phi_i^n(t,T_0)^{\mathrm{T}}\Lambda\phi_i^n(t,T_0)\exp\left(-2\int_{T_0}^{s} q^n(u)\,du\right).
\end{aligned} \tag{2.37}$$

Hence, for all $t \leq s \leq T$,

$$\phi_i^n(t,s)^{\mathrm{T}}\Lambda\phi_i^n(t,s) \leq \phi_i^n(t,t)^{\mathrm{T}}\Lambda\phi_i^n(t,t)\exp\left(-2\int_t^{s} q^n(u)\,du\right).$$

Since Λ is positive definite and $\phi_i^n(t,t)$ is the ith unit vector, it yields

$$\begin{aligned}
|\phi_i^n(t,s)|^2 &= \phi_i^n(t,s)^{\mathrm{T}} I_d\, \phi_i^n(t,s) \\
&\leq \frac{1}{\lambda_{\min}}\phi_i^n(t,s)^{\mathrm{T}}\Lambda\phi_i^n(t,s) \\
&\leq \frac{\lambda_{\max}}{\lambda_{\min}}\exp\left(-2\int_t^s q^n(u)\,du\right).
\end{aligned}$$

Hence,

$$|\Phi^n(t,s)|^2 = \sum_{1\leq i\leq d} |\phi_i^n(t,s)|^2 \leq d\frac{\lambda_{\max}}{\lambda_{\min}}\exp\left(-2\int_t^s q^n(u)\,du\right).$$

Since $\left|\Phi^n(t,s)^{-1}\right| = \left|\left(\Phi^n(t,s)^{-1}\right)^{\mathrm{T}}\right|$, we may consider the differential equation

$$\begin{cases} d\big(\Phi^n(t,s)^{-1}\big)^{\mathrm{T}} = (D^n(s) + E^n(s)\gamma)\Lambda^{-1}\left(\Phi^n(t,s)^{-1}\right)^{\mathrm{T}} ds, \\ \big(\Phi^n(t,t)^{-1}\big)^{\mathrm{T}} = I_d, \end{cases}$$

in order to establish the desired bound for the inverse. This system is similar to (2.34). The desired bounds thus follow from similar arguments as before. □

The following bounds on the state process $(X^{n,*}, Y^{n,*})$ are key to our subsequent analysis.

Proposition 2.3.6. *Let $n > n_0$ for n_0 as in (2.21). Then there exists a constant $C > 0$ that is independent of n, such that for all $s \in [t, T]$,*

$$\begin{aligned} |X^{n,*}(s)| &\leq C|\Phi^n(t,s)|, \\ |Y^{n,*}(s)| &\leq C. \end{aligned} \tag{2.38}$$

Proof. Let $\tilde{X}^{n,*}(s) = \Phi^n(t,s)^{-1}X^{n,*}(s)$. Differentiating this equation and using (2.33) yields

$$\begin{aligned} \tilde{X}^{n,*}(s) = x + \int_t^s \Phi^n(t,r)^{-1}\Lambda^{-1}E^n(r)\Big(&e^{-\int_t^r \rho(u)\,du}(y+\gamma x) \\ &+ \int_t^r e^{-\int_u^r \rho(v)\,dv}\gamma\rho(u)\Phi^n(t,u)\tilde{X}^{n,*}(u)\,du\Big)\,dr. \end{aligned}$$

Since $\rho \geq 0$,

$$\begin{aligned} |\tilde{X}^{n,*}(s)| \leq |x| + \int_t^s &|\Phi^n(t,r)^{-1}\Lambda^{-1}E^n(r)| \\ &\cdot \left(|y+\gamma x| + \int_t^r |\gamma\rho(u)\Phi^n(t,u)| \cdot |\tilde{X}^{n,*}(u)|\,du\right)\,dr. \end{aligned}$$

The integral version of Gronwall's inequality [BS92, Corollary 11.1] yields

$$\begin{aligned} |\tilde{X}^{n,*}_s| \leq &\left(|x| + \int_t^s |\Phi^n(t,r)^{-1}\Lambda^{-1}E^n(r)| \cdot |y+\gamma x|\,dr\right) \\ &\cdot \exp\left(\int_t^s |\Phi^n(t,r)^{-1}\Lambda^{-1}E^n(r)| \int_t^r |\gamma\rho(u)\Phi^n(t,u)|\,du\,dr\right). \end{aligned} \tag{2.39}$$

The process Φ^n is uniformly bounded as mentioned in (2.36) and ρ is essentially bounded on $[0,T]$. Moreover, we prove below that there exists a constant $C > 0$, which is independent of n such that for $s \in [t,T]$,

$$|E^n(s)| \leq C(T-s). \tag{2.40}$$

Then the desired bounds follow from Lemma 2.3.5 and Lemma A.2.1 as

$$\begin{aligned} &\sup_n \int_t^T |\Phi^n(t,r)^{-1}\Lambda^{-1}E^n(r)|\,dr \\ \leq &\int_t^T |\Phi^n(t,r)^{-1}| \cdot |\Lambda^{-1}| \cdot |E^n(r)|\,dr \\ \leq &\sup_n \int_t^T \sqrt{\frac{d\lambda_{\max}}{\lambda_{\min}}} \exp\left(\int_t^r p^n(u)\,du\right) \cdot |\Lambda^{-1}| \cdot C(T-r)\,dr \\ \leq &\sup_n |\Lambda^{-1}| \sqrt{\frac{d\lambda_{\max}}{\lambda_{\min}}} \left(\int_t^{T_0} \mathbb{I}_{t\in[0,T_0)} L \cdot C(T-r)\,dr + \int_{T_0}^T L \cdot C\,dr\right) \\ < &\infty. \end{aligned}$$

In order to establish the bound (2.40), we multiply $\begin{pmatrix}-I_d & \gamma\end{pmatrix}$ on the left and $\begin{pmatrix}0 \\ I_d\end{pmatrix}$ on the right in (2.10) and use the decomposition of the matrix Q introduced prior to Section 2.3.1. Thus,

$$\begin{cases} -dE^n(s) =\Big(- \big(D^n(s) + E^n(s)\gamma\big)\Lambda^{-1}E^n(s) - E^n(s)\rho(s) - \rho(s)\gamma C^n(s) \\ \qquad\qquad\qquad + 2\rho(s)\Big)\, ds - M_E^n(s)\, dW(s), \\ E^n(T) =0, \end{cases}$$

where $M_E^n := \begin{pmatrix}-I_d & \gamma\end{pmatrix} M^n \begin{pmatrix}0 \\ I_d\end{pmatrix} \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{d\times d})$. Recalling the definition of Φ^n in (2.34),

$$\begin{aligned} &d\left(\Phi^n(t,s)^{\mathrm{T}} E^n(s) e^{-\int_t^s \rho(u)\,du}\right) \\ =&\Phi^n(t,s)^{\mathrm{T}}\Big(dE^n(s) - \big(D^n(s) + E^n(s)\gamma\big)\Lambda^{-1}E^n(s) - E^n(s)\rho(s)\Big)e^{-\int_t^s \rho(u)\,du} \\ =&-\Phi^n(t,s)^{\mathrm{T}}\big(- \rho(s)\gamma C^n(s) + 2\rho(s)\big)e^{-\int_t^s \rho(u)\,du}\, ds \\ &+ \Phi^n(t,s)^{\mathrm{T}} M_E^n(s) e^{-\int_t^s \rho(u)\,du}\, dW(s). \end{aligned}$$

The uniform boundedness of Φ^n in (2.36) together with $M_E^n \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{d\times d})$ and $\rho \geq 0$ yields that $E^n(s)$ can be expressed by

$$\begin{aligned} &\mathbb{E}\Big[\big(\Phi^n(t,s)^{\mathrm{T}}\big)^{-1}\Phi^n(t,T)^{\mathrm{T}}E^n(T)e^{-\int_s^T \rho(u)\,du} \\ &\quad + \big(\Phi^n(t,s)^{\mathrm{T}}\big)^{-1}\int_s^T \Phi^n(t,r)^{\mathrm{T}}\big(- \rho(r)\gamma C^n(r) + 2\rho(r)\big)e^{-\int_s^r \rho(u)\,du}\, dr\Big|\mathcal{F}_s\Big]. \end{aligned}$$

It follows from the semigroup property of Φ^n that $\Phi^n(s,r)\Phi^n(t,s) = \Phi^n(t,r)$ for $t < s < r$. Thus we can obtain that

$$\big(\Phi^n(t,s)^{\mathrm{T}}\big)^{-1}\,\Phi^n(t,r)^{\mathrm{T}} = \Phi^n(s,r)^{\mathrm{T}}.$$

In view of (2.36), $|\Phi^n(s,r)^{\mathrm{T}}|$ is uniformly bounded with respect to (n,r,ω). Hence (2.40) follows from $E^n(T) = 0$ along with the boundedness of ρ and the uniform boundedness of the matrices C^n; cf. Lemma 2.3.1. $\square$

Theorem 2.2.1 ensures the admissibility of our feedback control $\xi^{n,*}$. Next, we are going to show that the feedback control $\xi^{n,*}$ given in (2.11) also satisfies

$$\xi^{n,*}\big(\cdot, X^{n,*}(\cdot), Y^{n,*}(\cdot)\big) \in L^\infty_{\mathcal{F}}(t,T;\mathbb{R}^d).$$

Proposition 2.3.7. *Let $n > n_0$ for n_0 as in* (2.21). *Then $\xi^{n,*}\big(\cdot, X^{n,*}(\cdot), Y^{n,*}(\cdot)\big)$ belongs to $L^\infty_{\mathcal{F}}(t,T;\mathbb{R}^d)$. In fact,*

$$\sup_{n,s}\xi^{n,*}\big(\cdot, X^{n,*}(\cdot), Y^{n,*}(\cdot)\big) < +\infty.$$

Proof. From (2.33), we know that for $s \in [t, T]$,

$$\begin{aligned}
&\xi^{n,*}\big(s, X^{n,*}(s), Y^{n,*}(s)\big)\\
=&\Lambda^{-1}\big(D^n(s) + E^n(s)\gamma\big)X^{n,*}(s) - \Lambda^{-1}E^n(s)e^{-\int_t^s \rho(r)dr}(y + \gamma x)\\
&- \Lambda^{-1}E^n(s)\int_t^s e^{-\int_u^s \rho(r)dr}\gamma\rho(u)X^{n,*}(u)\,du.
\end{aligned} \tag{2.41}$$

Since the uniform boundedness of portfolio processes and the processes E^n with respect to (n, s, ω) have already been obtained in Proposition 2.38 and (2.40), respectively, we just need to establish an L^∞-bound for the first term on the right side in (2.41).

To this end, we recall [Ber05, Theorem 8.4.9] that $|A|_{2,2} \leq |B|_{2,2}$ for any two symmetric matrices $0 \leq A \leq B$. Thus, by Proposition 2.3.4, Lemma 2.3.5, Proposition 2.3.6,

$$\begin{aligned}
&\sup_{n,s}\big|\Lambda^{-1}\big(D^n(s) + E^n(s)\gamma\big)X^{n,*}(s)\big|\\
\leq&\sup_{n,s}\big|\Lambda^{-1}\big(D^n(s) + E^n(s)\gamma\big)\big|_{2,2}|X^{n,*}(s)|\\
=&\sup_{n,s}|\sqrt{\Lambda^{-1}}F^n\sqrt{\Lambda^{-1}}|_{2,2}|X^{n,*}(s)|\\
\leq&\sup_{n,s}p^n(s)\cdot C|\Phi^n(t,s)|\\
\leq&\sup_{n,s}p^n(s)\cdot C\sqrt{\frac{d\lambda_{\max}}{\lambda_{\min}}}\exp\left(-\int_t^s q^n(u)\,du\right).
\end{aligned}$$

We claim that

$$\sup_{n,s}\big|\Lambda^{-1}\big(D^n(s) + E^n(s)\gamma\big)X^{n,*}(s)\big| < +\infty.$$

In fact, applying Lemma A.2.1,

$$\begin{aligned}
&\sup_{n,s\in[T_0,T]} p^n(s)\cdot\exp\left(-\int_t^s q^n(u)\,du\right)\\
\leq&\sup_{n,s\in[T_0,T]}\left(\frac{1}{T-s+\frac{1}{\frac{n-\gamma_{\min}}{\lambda_{\min}}-\sqrt{1+\alpha}-1}} + 1 + \sqrt{1+\alpha}\right)\\
&\qquad\cdot L\left(T-s+\frac{\lambda_{\max}}{n-\gamma_{\max}+\lambda_{\max}}\right)\\
<&+\infty.
\end{aligned}$$

For $s \in [t, T_0)$, the uniform boundedness can also be obtained by the estimates in Lemma A.2.1. Hence the desired uniform L^∞-bound is established. $\square$

2.3.3. Solving the optimal liquidation problem

The candidate value function

In this section, we prove that the limit

$$Q(t) := \lim_{n\to\infty} Q^n(t)$$

exists for $t \in [0,T)$. In particular, the candidate value function for (2.5),

$$\tilde{V}(t,x,y) := \begin{pmatrix} x^{\mathrm{T}} & y^{\mathrm{T}} \end{pmatrix} Q(t) \begin{pmatrix} x \\ y \end{pmatrix},$$

is well-defined.

Proof of Theorem 2.2.2. As discussed in Section 2.3.1, the sequence $\{Q^n(t)\}$ is non-decreasing for given $t \in [0,T)$. Moreover, the a priori estimates (2.19) imply that

$$|Q^n| \le \sqrt{|A^n_{\max}|^2 + |B^n_{\max}|^2 + |C^n_{\max}|^2} \le C$$

for some constant $C > 0$ uniformly on $[0,t]$. In particular, the sequence $\{Q^n(\cdot)\}$ converges pointwise and in L^2 to some limiting process $Q(\cdot)$ on $[0,t]$. Using the continuity of Q^n,

$$\liminf_{t\to T} |Q(t)| \ge \liminf_{t\to T} |Q(t)|_{2,2} \ge \liminf_{t\to T} |Q^n(t)|_{2,2} = |Q^n(T)|_{2,2} > n.$$

This shows that

$$\liminf_{t\to T} |Q(t)| = +\infty.$$

We are now going to show that Q is one part of the solution to the matrix differential equations (2.12) on $[0,T)$. To this end, let $n > m$, and let $(Q^n, M^n), (Q^m, M^m)$ be the solutions of (2.10) with terminal values $\begin{pmatrix} nI_d & I_d \\ I_d & \gamma^{-1} \end{pmatrix}$ and $\begin{pmatrix} mI_d & I_d \\ I_d & \gamma^{-1} \end{pmatrix}$, respectively. Applying the Itô formula to $|Q^n - Q^m|^2$ on $[s,t]$, we obtain

$$\begin{aligned} &|Q^n(s) - Q^m(s)|^2 + \int_s^t |M^n(r) - M^m(r)|^2\, dr \\ =&|Q^n(t) - Q^m(t)|^2 - 2\int_s^t \operatorname{tr}\Big(\big(Q^n(r) - Q^m(r)\big)\big(M^n(r) - M^m(r)\big)\Big)\, dW(r) \\ &+ 2\int_s^t \operatorname{tr}\bigg(\big(Q^n(r) - Q^m(r)\big)\Big(g\big(r, Q^n(r)\big) - g\big(r, Q^m(r)\big)\Big)\bigg)\, dr, \end{aligned} \tag{2.42}$$

where

$$\begin{aligned} g\big(r, Q(r)\big) :=& -Q(r)\begin{pmatrix} -I_d \\ \gamma \end{pmatrix} \Lambda^{-1} \begin{pmatrix} -I_d & \gamma \end{pmatrix} Q(r) + Q(r)\begin{pmatrix} 0 & 0 \\ 0 & -\rho(r) \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ 0 & -\rho(r) \end{pmatrix} Q(r) + \begin{pmatrix} \Sigma(r) & 0 \\ 0 & \gamma^{-1}\rho(r) + \rho(r)\gamma^{-1} \end{pmatrix} \end{aligned}$$

and

$$
\begin{aligned}
& g\big(r, Q^n(r)\big) - g\big(r, Q^m(r)\big) \\
=& -\left(Q^n(r) - Q^m(r)\right)\begin{pmatrix} -I_d \\ \gamma \end{pmatrix}\Lambda^{-1}\begin{pmatrix} -I_d & \gamma \end{pmatrix}\left(Q^n(r) - Q^m(r)\right) \\
& + \left(\begin{pmatrix} 0 & 0 \\ 0 & -\rho(r) \end{pmatrix} - Q^m(r)\begin{pmatrix} -I_d \\ \gamma \end{pmatrix}\Lambda^{-1}\begin{pmatrix} -I_d & \gamma \end{pmatrix}\right)\left(Q^n(r) - Q^m(r)\right) \\
& + \left(Q^n(r) - Q^m(r)\right)\left(\begin{pmatrix} 0 & 0 \\ 0 & -\rho(r) \end{pmatrix} - \begin{pmatrix} -I_d \\ \gamma \end{pmatrix}\Lambda^{-1}\begin{pmatrix} -I_d & \gamma \end{pmatrix} Q^m(r)\right).
\end{aligned}
$$

Due to the symmetry of $Q^n(r)$ and monotonicity of the sequence $\{Q^n(r)\}$, the square root $\sqrt{Q^n(r) - Q^m(r)}$ exists. Denote

$$
g_0^{n,m}(r) := \left(-\big(Q^n(r) - Q^m(r)\big)\begin{pmatrix} -I_d \\ \gamma \end{pmatrix}\Lambda^{-1}\begin{pmatrix} -I_d & \gamma \end{pmatrix}\left(Q^n(r) - Q^m(r)\right)\right).
$$

Since Λ^{-1} is positive definite,

$$
\begin{aligned}
& \operatorname{tr}\Big(\big(Q^n(r) - Q^m(r)\big) g_0^{n,m}(r)\Big) \\
=& -\operatorname{tr}\left(\left(Q^n(r) - Q^m(r)\right)^{\frac{3}{2}}\begin{pmatrix} -I_d \\ \gamma \end{pmatrix}\Lambda^{-1}\begin{pmatrix} -I_d & \gamma \end{pmatrix}\left(Q^n(r) - Q^m(r)\right)^{\frac{3}{2}}\right) \leq 0.
\end{aligned}
$$

Since $\rho, \Sigma \in L^\infty_{\mathcal{F}}(0, T; \mathcal{S}^d_+)$ and the sequence $\{Q^n\}$ is uniformly bounded on $[0, t]$,

$$
\sup_{0 \leq r \leq t}\left|\left(\begin{pmatrix} 0 & 0 \\ 0 & -\rho(r) \end{pmatrix} - \begin{pmatrix} -I_d \\ \gamma \end{pmatrix}\Lambda^{-1}\begin{pmatrix} -I_d & \gamma \end{pmatrix} Q^m(r)\right)\right| \leq C,
$$

for some constant $C > 0$ that is independent of n, m. Using $\operatorname{tr}(AB) \leq |A| \cdot |B|$,

$$
\operatorname{tr}\Big(\big(Q^n(r) - Q^m(r)\big)\Big(g\big(r, Q^n(r)\big) - g\big(r, Q^m(r)\big)\Big)\Big) \leq C|Q^n(r) - Q^m(r)|^2.
$$

Moreover, the fact that $M^n, M^m \in L^2_{\mathcal{F}}(0, T; \mathcal{S}^{2d})$ yields

$$
\mathbb{E}\left[\int_s^t \operatorname{tr}\Big(\big(Q^n(r) - Q^m(r)\big)\big(M^n(r) - M^m(r)\big)\Big)\, dW(r)\right] = 0.
$$

Hence,

$$
\begin{aligned}
& \mathbb{E}\left[\int_s^t |M^n(r) - M^m(r)|^2\, dr\right] \\
\leq & \mathbb{E}\left[|Q^n(t) - Q^m(t)|^2 + C\int_s^t |Q^n(r) - Q^m(r)|^2\, dr\right].
\end{aligned} \tag{2.43}
$$

Using the Burkholder-Davis-Gundy inequality in (2.42), we can find a constant $C > 0$ such that

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq s\leq t}|Q^n(s)-Q^m(s)|^2\right] \\
\leq&\mathbb{E}\left[|Q^n(t)-Q^m(t)|^2+\int_0^t C|Q^n(r)-Q^m(r)|^2\,dr\right] \\
&+C\mathbb{E}\left[\sqrt{\int_0^t |Q^n(r)-Q^m(r)|^2|M^n(r)-M^m(r)|^2\,dr}\right].
\end{aligned}
$$

By Young's inequality,

$$
\begin{aligned}
&C\mathbb{E}\left[\sqrt{\int_0^t |Q^n(r)-Q^m(r)|^2|M^n(r)-M^m(r)|^2\,dr}\right] \\
\leq&\frac{1}{2}\mathbb{E}\left[\sup_{0\leq s\leq t}|Q^n(s)-Q^m(s)|^2+C\mathbb{E}\int_0^t |M^n(r)-M^m(r)|^2\,dr\right].
\end{aligned}
$$

Altogether, we arrive at

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0s\leq t}|Q^n(s)-Q^m(s)|^2\,ds\right] \\
\leq&C\mathbb{E}\left[|Q^n(t)-Q^m(t)|^2+\int_0^t |Q^n(r)-Q^m(r)|^2dr\right].
\end{aligned}
$$

The right-hand side converges to zero as $n, m \to \infty$. This shows that

$$
Q \in L^\infty_{\mathcal{F}}\big(\Omega; C([0,T^-];\mathcal{S}^{2d}_+)\big).
$$

Furthermore, (2.43) implies that $\{M^n\}$ is a Cauchy sequence in $L^2_{\mathcal{F}}(0,t;\mathcal{S}^{2d})$ and converges to some $M \in L^2_{\mathcal{F}}(0,t;\mathcal{S}^{2d})$, for every $t < T$. Taking the limit $n \to \infty$ in (2.10) implies (Q, M) satisfies the matrix differential equations (2.12) on $[0,T)$. Compact convergence follows by Dini's theorem, due to the monotonicity. $\square$

Verification

Before proving that the strategy ξ^* defined in Theorem 2.2.3 is admissible, we first analyze the controlled processes X^*, Y^* defined as solutions to the equations

$$
\begin{cases}
dX^*(s) = -\xi^*\big(s, X^*(s), Y^*(s)\big)\,ds, \quad s\in[t,T), \\
dY^*(s) = \Big(-\rho(s)Y^*(s)+\gamma\xi^*\big(s,X^*(s),Y^*(s)\big)\Big)\,ds, \quad s\in[t,T), \\
\quad X(t)=x, \quad Y(t)=y
\end{cases}
$$

and show that ξ^* is a liquidation strategy, i.e. that $\lim_{s\to T} X^*(s) = 0$.

Proposition 2.3.8. *(i) Let* $Z^{n,*\mathrm{T}} = (X^{n,*\mathrm{T}}, Y^{n,*\mathrm{T}})$, $Z^{*\mathrm{T}} = (X^{*\mathrm{T}}, Y^{*\mathrm{T}})$. *Then*

$$Z^{n,*} \overset{n\to\infty}{\longrightarrow} Z^* \text{ compactly on } [t,T).$$

(ii) We have that

$$n|X^{n,*}(T)|^2 \overset{n\to\infty}{\longrightarrow} 0$$

and that

$$X^{n,*}(T) \overset{n\to\infty}{\longrightarrow} X^*(T) = \lim_{s\to T} X^*(s) = 0,$$
$$Y^{n,*}(T)^{\mathrm{T}} X^{n,*}(T) \overset{n\to\infty}{\longrightarrow} Y^*(T)^{\mathrm{T}} X^*(T) = 0.$$

Proof. (i) Let $t \le T' < T$. On $[t,T']$, Z^* and $Z^{n,*}$ solve the differential equations

$$dZ = \left(- \begin{pmatrix} -I_d \\ \gamma \end{pmatrix} \Lambda^{-1} \begin{pmatrix} -I_d & \gamma \end{pmatrix} R + \begin{pmatrix} 0 & 0 \\ 0 & -\rho \end{pmatrix} \right) Z\, dt$$

for $R = Q$ and $R = Q^n$, respectively. Since on $[0,T']$ the sequences $\{Q^n\}$ and $\{Z^{n,*}\}$ are uniformly bounded and $\{Q^n\}$ uniformly converges to Q, the first assertion follows from the continuous dependence of solutions of ordinary linear differential equations.

(ii) The convergence of the sequence $\{n|X^{n,*}(T)|^2\}$ to zero follows from Proposition 2.3.6 along with Lemma 2.3.5 and Lemma A.2.1. To be specific,

$$n|X^{n,*}(T)|^2 \le nC\frac{d\lambda_{max}}{\lambda_{min}} L^2 \left(\frac{\lambda_{max}}{n - \gamma_{max} + \lambda_{max}} \right)^2,$$

from which we see that

$$n|X^{n,*}(T)|^2 \overset{n\to\infty}{\longrightarrow} 0.$$

From the bound on $\sqrt{\Lambda^{-1}} F^n \sqrt{\Lambda^{-1}}$ in (2.30) and the definition of p^n, q^n in (2.29), (2.31), we know that

$$\lim_{n\to\infty} \sqrt{\Lambda^{-1}} F^n \sqrt{\Lambda^{-1}} = \sqrt{\Lambda^{-1}} F \sqrt{\Lambda^{-1}}$$

can be bounded by $p(s) := \lim_{n\to\infty} p^n(s)$ and $q(s) := \lim_{n\to\infty} q^n(s)$ from above and below, respectively.
Therefore, similar argument to the proof of Proposition 2.3.6 show that

$$|X^*(s)| \le C|\Phi(t,s)| \le C\sqrt{\frac{d\lambda_{\max}}{\lambda_{\min}}} e^{-\int_t^s q(u)du}.$$

By Lemma A.2.1, $\lim_{s\to T} e^{\int_t^s -q(u)du} = \lim_{s\to T} \lim_{n\to\infty} e^{\int_t^s -q^n(u)du} = 0$, which yields

$$\lim_{s\to T} X^*(s) = 0.$$

Using the uniform boundedness of $|Y^{n,*}|$ with respect (n,s,ω), similar arguments show that

$$\lim_{s\to T} Y^{n,*}(s)^{\mathrm{T}} X^{n,*}(s) = Y^*(T)^{\mathrm{T}} X^*(T) = 0.$$

$\square$

From the compact convergence results on $Q^n, X^{n,*}, Y^{n,*}$, we know that

$$\xi^{n,*}\big(\cdot, X^{n,*}(\cdot), Y^{n,*}(\cdot)\big) \overset{n\to\infty}{\longrightarrow} \xi^*\big(\cdot, X^*(\cdot), Y^*(\cdot)\big) \text{ compactly on } [t,T).$$

Proposition 2.3.9. *The feedback control ξ^* is an admissible liquidation strategy. That is,*

$$\xi^*\big(\cdot, X^*(\cdot), Y^*(\cdot)\big) \in \mathcal{A}(t,x,y).$$

Proof. For $s \in [t,T]$,

$$\begin{aligned}\xi^*\big(s, X^*(s), Y^*(s)\big) =& \Lambda^{-1}\big(D(s)+E(s)\gamma\big)X^*(s) - \Lambda^{-1}E(s)e^{-\int_t^s \rho(r)dr}(y+\gamma x)\\ &- \Lambda^{-1}E(s)\int_t^s e^{-\int_u^s \rho(r)dr}\gamma\rho(u)X^*(u)\,du.\end{aligned}$$

Similarly to the proof of Proposition 2.3.7, we have that

$$\begin{aligned}&\sup_s |\xi^*\big(s, X^*(s), Y^*(s)\big)|\\ \leq& \sup_s \Big\{|\Lambda^{-1}\big(D(s)+E(s)\gamma\big)X^*(s)| + |\Lambda^{-1}E(s)e^{-\int_t^s \rho(r)dr}(y+\gamma x)|\\ &+ |\Lambda^{-1}E(s)\int_t^s e^{-\int_u^s \rho(r)dr}\gamma\rho(u)X^*(u)\,du|\Big\}\\ <&\infty.\end{aligned}$$

In fact, it's worthy to notice that

$$\begin{aligned}&\sup_{s\in[T_0,T]} \big|\Lambda^{-1}(D(s)+E(s)\gamma)X^*(s)\big|\\ \leq& \sup_{s\in[T_0,T]} p(s)\cdot C\sqrt{\frac{d\lambda_{max}}{\lambda_{min}}}e^{-\int_t^s q(u)du}\\ \leq& \sup_{s\in[T_0,T]} C\sqrt{\frac{d\lambda_{max}}{\lambda_{min}}}\cdot\left(\frac{1}{T-s}+1+\sqrt{1+\alpha}\right)\cdot L(T-s)\\ <&\infty.\end{aligned}$$

Therefore,

$$\xi^*\big(\cdot, X^*(\cdot), Y^*(\cdot)\big) \in L^\infty_{\mathcal{F}}\big(\Omega; C([t,T];\mathbb{R}^d)\big).$$

□

We are now ready to verify that the limit of the solution to (2.7) is indeed the solution to the original problem (2.5).

Proof of Theorem 2.2.3. We fix $(t,x,y) \in [0,T)\times\mathbb{R}^d\times\mathbb{R}^d$. With a slight abuse of notation, we write $\xi^{n,*} = \xi^{n,*}(\cdot, X^{n,*}(\cdot), Y^{n,*}(\cdot)), \xi^* = \xi^*(\cdot, X^*(\cdot), Y^*(\cdot))$ in the following proof. Noticing that $\xi^{n,*}, X^{n,*}, Y^{n,*}$ are uniformly bounded with respect to (n, s, ω) and respectively converge to ξ^*, X^*, Y^* as $n \to +\infty$, we can apply the dominated convergence theorem to obtain

$$
\begin{aligned}
&\lim_{n\to\infty} \tilde{V}^n(t,x,y) \\
&= \lim_{n\to\infty} \mathbb{E}\Big[\frac{n}{2} X^{n,*}(T)^{\mathrm{T}} X^{n,*}(T) + Y^{n,*}(T)^{\mathrm{T}} X^{n,*}(T) + \frac{1}{2} Y^{n,*}(T)^{\mathrm{T}} \gamma^{-1} Y^{n,*}(T) \\
&\qquad\qquad + \int_t^T \frac{1}{2}\Big(\xi^{n,*}(s)^{\mathrm{T}} \Lambda \xi^{n,*}(s) + Y^{n,*}(s)^{\mathrm{T}} (\rho\gamma^{-1} + \gamma^{-1}\rho) Y^{n,*}(s) \\
&\qquad\qquad\qquad\qquad + X^{n,*}(s)^{\mathrm{T}} \Sigma(s) X^{n,*}(s)\Big)\, ds \Big| \mathcal{F}_t\Big] \\
&= \mathbb{E}\Big[\lim_{n\to\infty} \Big(\frac{n}{2} X^{n,*}(T)^{\mathrm{T}} X^{n,*}(T) + Y^{n,*}(T)^{\mathrm{T}} X^{n,*}(T) \\
&\qquad\qquad + \frac{1}{2} Y^{n,*}(T)^{\mathrm{T}} \gamma^{-1} Y^{n,*}(T) + \int_t^T \frac{1}{2}\big(\xi^{n,*}(s)^{\mathrm{T}} \Lambda \xi^{n,*}(s) \\
&\qquad\qquad + Y^{n,*}(s)^{\mathrm{T}} (\rho\gamma^{-1} + \gamma^{-1}\rho) Y^{n,*}(s) X^{n,*}(s)^{\mathrm{T}} \Sigma(s) X^{n,*}(s)\big)\, ds\Big) \Big| \mathcal{F}_t\Big] \\
&= \mathbb{E}\Big[\int_t^T \lim_{n\to\infty} \frac{1}{2}\big(\xi^{n,*}(s)^{\mathrm{T}} \Lambda \xi^{n,*}(s) + Y^{n,*}(s)^{\mathrm{T}} (\rho\gamma^{-1} + \gamma^{-1}\rho) Y^{n,*}(s) \\
&\qquad\qquad\qquad + X^{n,*}(s)^{\mathrm{T}} \Sigma(s) X^{n,*}(s)\big)\, ds + \frac{1}{2} Y(T)^{\mathrm{T}} \gamma^{-1} Y(T) \Big| \mathcal{F}_t\Big] \\
&= \mathbb{E}\Big[\int_t^T \frac{1}{2}\big(\xi^*(s)^{\mathrm{T}} \Lambda \xi^*(s) + Y^*(s)^{\mathrm{T}} (\rho\gamma^{-1} + \gamma^{-1}\rho) Y^*(s) \\
&\qquad\qquad + X^*(s)^{\mathrm{T}} \Sigma(s) X^*(s)\big)\, ds + \frac{1}{2} Y(T)^{\mathrm{T}} \gamma^{-1} Y(T) \Big| \mathcal{F}_t\Big] \\
&\geq \tilde{V}(t,x,y)
\end{aligned}
$$

Since we already have that $\tilde{V}^n(t,x,y) \leq \tilde{V}(t,x,y)$, the equality follows. Therefore, ξ^* solves the Optimization Problem (2.5) and the value function is given by $\tilde{V}$. □

3. Continuous viscosity solutions to portfolio liquidation problems

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ that satisfies the usual conditions and carries a Poisson process N and an independent $\tilde{d}$-dimensional standard Brownian motion W. In this Chapter, we consider a portfolio liquidation problem under price-sensitive market impact. This problem leads to the following stochastic control problem:

$$\operatorname*{ess\,inf}_{\xi,\mu} E\left[\int_0^T \eta(Y_s)|\xi_s|^2 + \theta\gamma(Y_s)|\mu_s|^2 + \lambda(Y_s)|X_s^{\xi,\mu}|^2\,ds\right] \tag{3.1}$$

subject to the state dynamics

$$\begin{aligned} dY_t &= b(Y_t)dt + \sigma(Y_t)dW_t, \quad Y_0 = y \\ dX_t^{\xi,\mu} &= -\xi_t\,dt - \mu_t\,dN_t, \quad X_0^{\xi,\mu} = x \end{aligned}$$

and the terminal state constraint

$$X_T^{\xi,\mu} = 0.$$

This Chapter is organized as follows. In Section 3.1, we summarize our main results. The existence of a unique viscosity solution to the HJB equation is proved in Section 3.2.1; the verification argument is carried out in Section 3.2.2. Section 3.3 is devoted to an extension of our uniqueness result to a non-Markovian model with unbounded coefficients.

3.1. Assumptions and main results

For each initial state $(t, y, x) \in [0, T) \times \mathbb{R}^d \times \mathbb{R}$ we define by

$$V(t,y,x) := \inf_{(\xi,\mu)\in\mathcal{A}(t,x)} E\left[\int_t^T \eta(Y_s^{t,y})|\xi_s|^2 + \theta\gamma(Y_s^{t,y})|\mu_s|^2 + \lambda(Y_s^{t,y})|X_s^{\xi,\mu}|^2\,ds\right] \tag{3.2}$$

the *value function* of the control problem (3.1) subject to the state dynamics

$$\begin{aligned} dY_s^{t,y} &= b(Y_s^{t,y})ds + \sigma(Y_s^{t,y})dW_s, \quad Y_t^{t,y} = y \\ dX_s^{\xi,\mu} &= -\xi_s\,ds - \mu_s\,dN_s, \quad X_t = x. \end{aligned} \tag{3.3}$$

Here, $\xi = (\xi_s)_{s\in[t,T]}$ describes the *rates* at which the agent trades in the primary market, while $\mu = (\mu_s)_{s\in[t,T]}$ describes the *orders* submitted to the dark pool. The infimum is taken over the set $\mathcal{A}(t,x)$ of all *admissible controls*, that is, over all pairs of controls (ξ,μ) such that $\xi \in L^4_{\mathcal{F}}(t,T;\mathbb{R})$, such that μ is predictable[1] and such that the resulting state process

$$X_s^{\xi,\mu} = x - \int_t^s \xi_r\,dr - \int_t^s \mu_r\,dN_r, \qquad t \le s \le T,$$

satisfies the terminal state constraint

$$X_T^{\xi,\mu} = 0. \tag{3.4}$$

The expected costs associated with an admissible liquidation strategy (ξ,μ) are given by

$$J(t,y,x;\xi,\mu) := \mathbb{E}\left[\int_t^T c(Y_s^{t,y}, X_s^{\xi,\mu}, \xi_s, \mu_s)\,ds\right],$$

where the running cost function $c(y,x,\xi,\mu)$ is given by

$$c(y,x,\xi,\mu) := \eta(y)|\xi|^2 + \theta\gamma(y)|\mu|^2 + \lambda(y)|x|^2.$$

Remark 3.1.1. We assume that the cost function is quadratic in the controls and the state variable. A generalization to general powers $p > 1$ as in [GHS18] can be established using similar arguments but renders the notation more cumbersome.

The dynamic programming principle suggests that the value function satisfies the HJB equation

$$-\partial_t V(t,y,x) - \mathcal{L}V(t,y,x) - \inf_{\xi,\mu\in\mathbb{R}} H(t,y,x,\xi,\mu,V) = 0, \quad (t,y,x) \in [0,T)\times\mathbb{R}^d\times\mathbb{R}, \tag{3.5}$$

where

$$\mathcal{L} := \frac{1}{2}\operatorname{tr}(\sigma\sigma^* D_y^2) + \langle b, D_y\rangle$$

denotes the infinitesimal generator of the factor process and the Hamiltonian H is given by

$$H(t,y,x,\xi,\mu,V) := -\xi\partial_x V(t,y,x) + \theta(V(t,y,x-\mu) - V(t,y,x)) + c(y,x,\xi,\mu).$$

The quadratic cost function suggests an ansatz of the form $V(t,y,x) = v(t,y)|x|^2$. The following result confirms this intuition. Its proof can be found in [GHS18, Section 2.2].

Lemma 3.1.2. *A nonnegative function $v : [0,T)\times\mathbb{R}^d \to [0,\infty)$ is a (sub/super) solution to the PDE*

$$-\partial_t v(t,y) - \mathcal{L}v(t,y) - F(y, v(t,y)) = 0, \tag{3.6}$$

where

$$F(y,v) := \lambda(y) - \frac{|v|^2}{\eta(y)} + \frac{\theta\gamma(y)v}{\gamma(y)+|v|} - \theta v, \tag{3.7}$$

[1] We show later that we restrict ourselves to monotone portfolio processes so we could just as well assume that μ is bounded.

if and only if $v(t,y)|x|^2$ *is a (sub/super) solution to the HJB equation* (3.5)*. In this case the infimum in* (3.5) *is attained at*

$$\xi^*(t,y,x) = \frac{v(t,y)}{\eta(y)}x \quad \textit{and} \quad \mu^*(t,y,x) = \frac{v(t,y)}{\gamma(y)+v(t,y)}x \tag{3.8}$$

and

$$H(t,y,x,\xi^*(t,y,x),\mu^*(t,y,x),v(\cdot,\cdot)|\cdot|^2) = F(y,v(t,y))|x|^2. \tag{3.9}$$

3.1.1. Assumptions

In order to prove the existence of a unique non-negative continuous viscosity solution of polynomial growth to our HJB equation we assume throughout that the factor process

$$Y_s^{t,y} = y + \int_t^s b(Y_r^{t,y})\,dr + \int_t^s \sigma(Y_r^{t,y})\,dW_r, \qquad t \le s \le T. \tag{3.10}$$

satisfies the following condition.

Assumption 3.1.3. The coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times\tilde{d}}$ are Lipschitz continuous.

The preceding assumption guarantees that the SDE (3.10) has a unique strong solution $(Y_s^{t,y})_{s\in[t,T]}$ for every initial state $(t,y) \in [0,T]\times\mathbb{R}^d$ and that the mapping $(s,t,y) \mapsto Y_s^{t,y}$ is a.s. continuous. We repeatedly use the following well known estimates; cf. [Kry80, Corollary 2.5.12]. For all $m \ge 0$, there exists $C > 0$ such that for all $y \in \mathbb{R}^d, 0 \le t \le s \le T$,

$$\mathbb{E} \sup_{t\le s\le T} |Y_s^{t,y}|^m \le C(1+|y|^m). \tag{3.11}$$

Furthermore, we assume that the cost coefficients are continuous and of polynomial growth and that η is twice continuously differentiable and satisfies a mild boundedness condition.

Assumption 3.1.4. The cost coefficients satisfy the following conditions:

(i) The coefficients $\eta, \gamma, \lambda, 1/\eta : \mathbb{R}^d \to [0,\infty)$ are continuous and of polynomial growth.

(ii) $\eta \in C^2$ and $\|\frac{\mathcal{L}\eta}{\eta}\|$ is bounded.

Remark 3.1.5. The preceding assumption is satisfied if, for instance Y is a geometric Brownian motion or an Ornstein-Uhlenbeck (OU) process and

$$\eta(y) = 1 + |y|^2.$$

In both cases, condition (2.13) in [Sch13a] is violated. Our assumptions are also weaker than those in [GHS18]. For instance, OU processes do not generate analytic semigroups, they do not satisfy the assumptions therein.

3.1.2. Main results

Before stating our first main result, we recall the notion of viscosity solutions for parabolic equations that will be used in this paper. The following definition can be found in [CIL92, Section 8].

Definition 3.1.6. For semicontinuous functions $v : [0,T) \times \mathbb{R}^d \to \mathbb{R}$ we use the following solution concepts for the parabolic PDE:

$$-\partial_t v(t,y) - G(t,y,v(t,y),D_y v(t,y),D_y^2 v(t,y)) = 0, \tag{3.12}$$

where $G : [0,T) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}$ and $\mathbb{S}^d$ denotes the set of symmetric $d \times d$ matrices.

(i) A function $v \in USC_m([0,T^-] \times \mathbb{R}^d)$ is a *(strict) viscosity subsolution* if for every $\varphi \in C^{1,2}_{loc}([0,T) \times \mathbb{R}^d)$ such that $\varphi \geq v$ and $\varphi(t,y) = v(t,y)$ at a point $(t,y) \in [0,T) \times \mathbb{R}^d$ it holds

$$-\partial_t \varphi(t,y) - G(t,y,v(t,y),D_y\varphi(t,y),D_y^2\varphi(t,y))(<) \leq 0.$$

(ii) A function $v \in LSC_m([0,T^-] \times \mathbb{R}^d)$ is a *(strict) viscosity supersolution* if for every $\varphi \in C^{1,2}_{loc}([0,T) \times \mathbb{R}^d)$ such that $\varphi \leq v$ and $\varphi(t,y) = v(t,y)$ at a point $(t,y) \in [0,T) \times \mathbb{R}^d$ it holds

$$-\partial_t \varphi(t,y) - G(t,y,v(t,y),D_y\varphi(t,y),D_y^2\varphi(t,y))(>) \geq 0.$$

(iii) A function v is a *viscosity solution* if v is both viscosity sub- and supersolution.

We are now ready to state the main result of this paper. Its proof is given in Section 3.2 below.

Theorem 3.1.7. *Under Assumptions 3.1.3, 3.1.4, the singular terminal value problem*

$$\begin{cases} -\partial_t v(t,y) - \mathcal{L}v(t,y) - F(y,v(t,y)) = 0, & (t,y) \in [0,T) \times \mathbb{R}^d, \\ \lim\limits_{t\to T} v(t,y) = +\infty & \textit{locally uniformly on } \mathbb{R}^d, \end{cases} \tag{3.13}$$

with the nonlinearity F given in (3.7) *admits a unique nonnegative viscosity solution in*

$$C_m([0,T^-] \times \mathbb{R}^d)$$

for some $m \geq 0$.

The next result states that both the value function and the optimal controls are given in terms of the unique viscosity solution to the HJB equation. The particular form of the feedback has been established in the literature before. What the proposition shows is that having a continuous viscosity solution to the HJB equation is enough to carry out the verification argument.

Proposition 3.1.8. *Under Assumptions 3.1.3 ,3.1.4, let v be the unique nonnegative viscosity solution to the singular terminal value problem* (3.13)*. Then, the value function* (3.2) *is given by $V(t,y,x) = v(t,y)|x|^2$, and the optimal control (ξ^*, μ^*) is given in feedback form by*

$$\xi^*_s = \frac{v(s, Y^{t,y}_s)}{\eta(Y^{t,y}_s)} X^*_s \quad \text{and} \quad \mu^*_s = \frac{v(s, Y^{t,y}_s)}{\gamma(Y^{t,y}_s) + v(s, Y^{t,y}_s)} X^*_{s-}. \tag{3.14}$$

In particular, the resulting optimal portfolio process $(X^*_s)_{s\in[t,T]}$ is given by

$$X^*_s = x \exp\left(-\int_t^s \frac{v(r, Y^{t,y}_r)}{\eta(Y^{t,y}_r)} dr \right) \prod_{t<r\le s}^{\Delta N_r \neq 0} \left(1 - \frac{v(t, Y^{t,y}_r)}{\gamma(Y^{t,y}_r) + v(t, Y^{t,y}_r)} \right). \tag{3.15}$$

Let us close this section with a model of optimal portfolio liquidation where market impact is driven by an Ornstein-Uhlenbeck process while market risk is driven by a geometric Brownian motion. Specifically, let $Y = (Y^1, Y^2)$ be the diffusion process given by

$$dY^1_t = -Y^1_t dt + dW^1_t \quad \text{and} \quad \frac{dY^2_t}{Y^2_t} = \sigma dW^2_t,$$

where W^1 and W^2 are two (possibly correlated) Brownian motions, and let

$$\eta(Y) = \begin{cases} 1 + |Y^1|^2, & \text{if } Y^1 < 0, \\ \dfrac{1}{1 + |Y^1|^2}, & \text{if } Y^1 \ge 0, \end{cases} \quad \gamma(Y) = 1, \quad \text{and} \quad \lambda(Y) = \sigma^2 |Y^2|^2.$$

The process Y^1 specifies a liquidity indicator that fluctuates around a stationary level (normalized to zero) with the market impact increasing when below average liquidity is available and decreasing when above average liquidity is available. Instantaneous market risk, on the other hand is captured by the volatility of the portfolio value assuming that asset prices follow a geometric Brownian motion. For the above choice of model parameters all assumptions on the cost and diffusion coefficients are satisfied. Hence, there exists a unique optimal liquidation strategy.

Remark 3.1.9. To the best of our knowledge, numerical methods for simulating solutions to general PDEs with singular terminal values are still to be developed. At least two problems arise when simulating solutions to HJB equations with singular terminal state constraint. The most obvious problem is the singular terminal condition. This problem can potentially be overcome by noting that the function

$$w(t,y) := (T-t)v(t,y), \quad (t,y) \in [0,T) \times \mathbb{R}$$

satisfies the following PDE with finite terminal value, yet singular driver (see [GHS18, GP19] and Section 3.2 for details)

$$\begin{cases} -\partial_t w(t,y) - \mathcal{L}w(t,y) - \dfrac{w(t,y)}{T-t} - (T-t)F(y, \dfrac{w(t,y)}{T-t}) = 0, & (t,y) \in [0,T) \times \mathbb{R}^d, \\ \lim\limits_{t\to T} w(t,y) = \eta(y) & \text{on } \mathbb{R}^d. \end{cases}$$

The knowledge of a unique classical solution to the transformed problem opens up the possibility to apply higher-order numerical schemes and obtain accurate solutions in acceptable computing time. One possibility could be to study a one-to-one mapping of the unbounded control set to a compact set combined with a discretisation of the control, similar to the idea applied to an optimal investment problem in [RF16]; an alternative approach based on monotonicity arguments is outlined in [GP19]. The second problem is to fix appropriate boundary conditions (in space) for the numerical simulations; a similar problem arises if the binding state constraint is replaced by a finite penalty term. The analysis in Section 3.2 shows that for the benchmark case of a risk neutral investor ($\lambda = 0$),

$$w(t, y) \leq C\eta(y), \quad (t, y) \in [0, T) \times \mathbb{R}$$

for some $C > 0$ from which we deduce zero boundary conditions if $\eta(y) \to 0$ for $|y| \to \infty$. In general we can not expect the above inequality to be an equality, though, not even asymptotically when $|y| \to \infty$. If we choose $\sigma = 0$ and the dynamics

$$dY_t = -\tanh(Y_t - Y_t^3)dt + dW_t$$

for the liquidity index, then the index is mean-reverting to the levels ± 1, the "regimes of average liquidity'. Choosing $\eta(y) = \frac{1}{1+y^2}$ all our assumptions on the model parameters are satisfied. In this case we may regard the interval $(-1, +1)$ as the low and the set $[-1, 1]^c$ as the high liquidity regime. Since $w(t, y) \to 0$ as $|y| \to \infty$, the boundary problem can be dealt with.

3.2. Solution and verification

3.2.1. Existence of solutions

In this section, we prove Theorem 3.1.7. In a first step, we establish a comparison principle for semicontinuous viscosity solutions to (3.13). In view of the singular terminal state constraint we can not follow the usual approach of showing that if a l.s.c. supersolution dominates an u.s.c. subsolution at the boundary, then it also dominates the subsolution on the entire domain. Instead, we prove that if some form of asymptotic dominance holds at the terminal time, then dominance holds near the terminal time.

In a second step, we construct smooth sub- and supersolutions to (3.13) that satisfy the required asymptotic dominance condition. Subsequently, we apply Perron's method to establish an u.s.c. subsolution and a l.s.c. supersolution that are bounded from above/below by the smooth solutions. From this, we infer that the semi-continuous solutions can be applied to the comparison principle, which then implies the existence of the desired continuous viscosity solution.

Comparison principle

Throughout this section, we fix $\delta \in (0, T]$ and $m \geq 0$, let $\overline{u} \in LSC_m([T-\delta, T^-] \times \mathbb{R}^d)$ and $\underline{u} \in USC_m([T-\delta, T^-] \times \mathbb{R}^d)$ be a viscosity super- and a viscosity subsolution to (3.13).

Proposition 3.2.1. *Under Assumptions 3.1.3, 3.1.4, if, uniformly on $\mathbb{R}^d$,*

$$\limsup_{t \to T} \frac{\underline{u}(t,y)(T-t) - \eta(y)}{1 + |y|^m} \leq 0 \leq \liminf_{t \to T} \frac{\overline{u}(t,y)(T-t) - \eta(y)}{1 + |y|^m}, \tag{3.16}$$

and

$$\underline{u}(t,y)(T-t), \overline{u}(t,y)(T-t) \geq \frac{1}{2}\eta(y), \quad t \in [T-\delta, T), \tag{3.17}$$

then

$$\underline{u} \leq \overline{u} \quad \textit{on} \quad [T-\delta, T) \times \mathbb{R}^d.$$

Assumptions (3.16), (3.17) are uncommon in the viscosity literature. However, we shall only use the comparison result to establish the existence of a solution, not the uniqueness. As a result, we only need to guarantee that the semi-continuous solutions established through Perron's method satisfy both assumptions.

The proof of the comparison principle is based on three auxiliary results. The first lemma is taken from [GHS18, Lemma A.2]. It is a modification of [BBP97, Lemma 3.7].

Lemma 3.2.2. *The difference $w := \underline{u} - \overline{u} \in USC_m([T-\delta, T^-] \times \mathbb{R}^d)$ is a viscosity subsolution to*

$$-\partial_t w(t,y) - \mathcal{L}w(t,y) - l(t,y)w(t,y) = 0, \quad (t,y) \in [T-\delta, T) \times \mathbb{R}^d, \tag{3.18}$$

where

$$l(t,y) := \frac{F(y, \underline{u}(t,y)) - F(y, \overline{u}(t,y))}{\underline{u}(t,y) - \overline{u}(t,y)} \mathbb{I}_{\underline{u}(t,y) \neq \overline{u}(t,y)}.$$

The next lemma constructs a smooth strict supersolution to (3.18) of polynomial growth.

Lemma 3.2.3. *For every $n \in \mathbb{N}$, there exists K_n large enough such that*

$$\chi(t,y) := \frac{e^{K_n(T-t)}(1+|y|^2)^{\frac{n}{2}}}{T-t}$$

satisfies

$$-\partial_t \chi(t,y) - \mathcal{L}\chi(t,y) + \frac{\chi(t,y)}{T-t} > 0, \quad (t,y) \in [T-\delta, T) \times \mathbb{R}^d.$$

Proof. Direct calculations verify that $h(t,y) := e^{K_n(T-t)}(1+|y|^2)^{\frac{n}{2}}$ satisfies

$$-\partial_t h(t,y) - \mathcal{L}h(t,y) > 0$$

in $[T-\delta,T)\times\mathbb{R}^d$ when K_n is chosen sufficiently large; see also [AT96, Proposition 5]. Here it is used that b and σ are Lipschitz and thus are of linear growth. Hence,

$$-\partial_t\chi(t,y)-\mathcal{L}\chi(t,y)+\frac{\chi(t,y)}{T-t}=\frac{-\partial_t h(t,y)-\mathcal{L}h(t,y)}{T-t}>0.$$

□

The following lemma is key to the proof of the comparison principle.

Lemma 3.2.4. *If $n\in\mathbb{N}$ in Lemma 3.2.3 is chosen large enough, then independent of $\alpha>0$, the function*

$$\Phi_\alpha(t,y):=w(t,y)-\alpha\chi(t,y)$$

is either nonpositive or attains its supremum at some point (t_α,y_α) in $[T-\delta,T)\times\mathbb{R}^d$.

Proof. Suppose that the supremum of Φ_α on $[T-\delta,T)\times\mathbb{R}^d$ is positive and denote by (t_k,y_k) a sequence in $[T-\delta,T)\times\mathbb{R}^d$ approaching the supremum point. The representation

$$\Phi_\alpha(t,y)=\frac{\left[\frac{\underline{u}(t,y)(T-t)-\eta(y)}{1+|y|^m}-\frac{\overline{u}(t,y)(T-t)-\eta(y)}{1+|y|^m}\right](1+|y|^m)-\alpha e^{K_n(T-t)}(1+|y|^2)^{\frac{n}{2}}}{T-t},$$

along with condition (3.16) shows that for any $n>m$,

$$\limsup_{t\to T}\Phi_\alpha(t,y)=-\infty,\ \text{uniformly on } \mathbb{R}^d.$$

Hence $\lim_k t_k<T$. Furthermore, $w\in USC_m([T-\delta,T^-]\times\mathbb{R}^d)$ is bounded by a function of polynomial growth uniformly away from the terminal time. Choosing n large enough this shows that $\lim_k|y_k|<\infty$. As a result, the supremum is attained at some point (t_α,y_α) because Φ_α is upper semicontinuous. This proves the assertion. □

We are now ready to prove the comparison principle.

Proof of Proposition 3.2.1. Let us fix $\alpha>0$. By letting $\alpha\to 0$ it is sufficient to show that the function Φ_α is nonpositive.

In view of Lemma 3.2.4, we just need to consider the case where there exists a point $(t_\alpha,y_\alpha)\in[T-\delta,T)\times\mathbb{R}^d$ such that

$$w(t,y)-\alpha\chi(t,y)\le w(t_\alpha,y_\alpha)-\alpha\chi(t_\alpha,y_\alpha),\quad (t,y)\in[T-\delta,T)\times\mathbb{R}^d.$$

This inequality can be interpreted as $w-\psi_\alpha$ having a global maximum at (t_α,y_α), where

$$\psi_\alpha:=\alpha\chi(t,y)+(w-\alpha\chi)(t_\alpha,y_\alpha).$$

Since ψ_α is smooth and w is a viscosity subsolution to (3.18),

$$-\partial_t\psi_\alpha(t_\alpha,y_\alpha)-\mathcal{L}\psi_\alpha(t_\alpha,y_\alpha)-l(t_\alpha,y_\alpha)w(t_\alpha,y_\alpha)\le 0.$$

By the mean value theorem along with the monotonicity of $\partial_u F$ and condition (3.17)

$$l(t,y) = \frac{F(y,\underline{u}(t,y)) - F(y,\overline{u}(t,y))}{\underline{u}(t,y) - \overline{u}(t,y)} \mathbb{I}_{\underline{u}(t,y)\neq\overline{u}(t,y)} \leq \partial_u F(y, \frac{\eta(y)}{2(T-t)}) = -\frac{1}{T-t}.$$

Thus, Lemma 3.2.3 implies

$$\begin{aligned} 0 \geq & -\partial_t \psi_\alpha(t_\alpha,y_\alpha) - \mathcal{L}\psi_\alpha(t_\alpha,y_\alpha) - l(t_\alpha,y_\alpha)w(t_\alpha,y_\alpha) \\ =& \alpha[-\partial_t \chi(t_\alpha,y_\alpha) - \mathcal{L}\chi(t_\alpha,y_\alpha) - l(t_\alpha,y_\alpha)w(t_\alpha,y_\alpha)] \\ >& -\alpha\frac{\chi(t_\alpha,y_\alpha)}{T-t_\alpha} - l(t_\alpha,y_\alpha)w(t_\alpha,y_\alpha) \\ \geq& \alpha l(t_\alpha,y_\alpha)\chi(t_\alpha,y_\alpha) - l(t_\alpha,y_\alpha)w(t_\alpha,y_\alpha) \\ =& -l(t_\alpha,y_\alpha)\Phi_\alpha(t_\alpha,y_\alpha). \end{aligned} \tag{3.19}$$

Since $l \leq 0$, we can conclude that $\Phi_\alpha(t_\alpha,y_\alpha) \leq 0$, thus $\Phi_\alpha \leq 0$. □

Existence via Perron's method

Armed with our comparison principle, the existence of a viscosity solution to our HJB equation can be established using Perron's method as soon as suitable sub- and supersolutions can be identified. In view of Assumption 3.1.4, $\eta, \lambda \in C_m(\mathbb{R}^d)$ for some $m \geq 0$ and $\|\frac{\mathcal{L}\eta}{\eta}\|$ is well-defined and finite. Hence

$$\delta := 1/\|\frac{\mathcal{L}\eta}{\eta}\| \wedge T > 0.^2 \tag{3.20}$$

By a direct computation, we can find a constant K' large enough such that the function: $\hat{h}(t,y) := e^{K'(T-t)}(1+|y|^2)^{m/2}$ satisfying

$$-\partial_t \hat{h}(t,y) - \mathcal{L}\hat{h}(t,y) - \lambda(y) \geq 0.$$

Let us then define

$$\check{v}(t,y) := \frac{\eta(y) - \eta(y)\|\frac{\mathcal{L}\eta}{\eta}\|(T-t)}{e^{\theta(T-t)}(T-t)} \quad \text{and} \quad \hat{v}(t,y) := \frac{\eta(y) + \eta(y)\|\frac{\mathcal{L}\eta}{\eta}\|(T-t)}{(T-t)} + \hat{h}(t,y).$$

Proposition 3.2.5. *Under Assumption 3.1.3, 3.1.4 the functions $\check{v}, \hat{v}$ are a non-negative classical sub- and supersolution to* (3.13) *on* $[T-\delta, T) \times \mathbb{R}^d$*, respectively.*

Proof. To verify the supersolution property of $\hat{v}$, we first verify that

$$\begin{aligned} & -\partial_t \hat{v}(t,y) - \mathcal{L}\hat{v}(t,y) \\ =& -\frac{\eta(y) + \mathcal{L}\eta(y)(T-t) + \mathcal{L}\eta(y)\|\frac{\mathcal{L}\eta}{\eta}\|(T-t)^2}{(T-t)^2} - \partial_t \hat{h}(t,y) - \mathcal{L}\hat{h}(t,y) \end{aligned} \tag{3.21}$$

[2] We use the convention $1/0 = \infty$.

Recalling the definition (3.7) of F, we have since $\hat{v} \geq 0$,

$$-F(y, \hat{v}(t,y)) \geq -\lambda(y) + \frac{\hat{v}(t,y)^2}{\eta(y)}.$$

Next, we apply the inequality $(u+v+w)^2 \geq u^2 + 2uv$ for $u, v, w \geq 0$ to the term $\hat{v}(t,y)^2$ to obtain

$$-F(y, \hat{v}(t,y)) \geq -\lambda(y) + \frac{\eta(y)^2 + 2\eta(y)^2 \|\frac{\mathcal{L}\eta}{\eta}\|(T-t)}{\eta(y)(T-t)^2}. \tag{3.22}$$

Adding (3.21) and (3.22) yields

$$\begin{aligned}-\partial_t \hat{v}(t,y) - \mathcal{L}\hat{v}(t,y) - F(y, \hat{v}(t,y)) \geq & \frac{2\eta(y)\|\frac{\mathcal{L}\eta}{\eta}\| - \mathcal{L}\eta(y) - \mathcal{L}\eta(y)\|\frac{\mathcal{L}\eta}{\eta}\|(T-t)}{(T-t)} \\ & - \partial_t \hat{h}(t,y) - \mathcal{L}\hat{h}(t,y) - \lambda(y).\end{aligned}$$

The definition of δ yields $1 \geq \|\frac{\mathcal{L}\eta}{\eta}\|(T-t)$ for $t \in [T-\delta, T)$ and so,

$$\begin{aligned}& 2\eta(y)\|\frac{\mathcal{L}\eta}{\eta}\| - \mathcal{L}\eta(y) - \mathcal{L}\eta(y)\|\frac{\mathcal{L}\eta}{\eta}\|(T-t) \\ \geq & \eta(y)\|\frac{\mathcal{L}\eta}{\eta}\| \cdot \left[1 + \|\frac{\mathcal{L}\eta}{\eta}\|(T-t)\right] - \mathcal{L}\eta(y) - \mathcal{L}\eta(y)\|\frac{\mathcal{L}\eta}{\eta}\|(T-t) \\ = & \left[1 + \|\frac{\mathcal{L}\eta}{\eta}\|(T-t)\right] \cdot \left[\eta(y)\|\frac{\mathcal{L}\eta}{\eta}\| - \mathcal{L}\eta(y)\right] \geq 0.\end{aligned}$$

We conclude that

$$-\partial_t \hat{v}(t,y) - \mathcal{L}\hat{v}(t,y) - F(y, \hat{v}(t,y)) \geq 0.$$

Next, we verify the subsolution property of $\check{v}$. By direct computation,

$$-\partial_t \check{v}(t,y) - \mathcal{L}\check{v}(t,y) = -\frac{\eta(y) + \mathcal{L}\eta(y)(T-t) - \mathcal{L}\eta(y)\|\frac{\mathcal{L}\eta}{\eta}\|(T-t)^2}{e^{\theta(T-t)}(T-t)^2} - \theta\check{v}(t,y). \tag{3.23}$$

On the other hand, since $\lambda, \gamma \geq 0$, and $\check{v} \geq 0$ on $[T-\delta, T) \times \mathbb{R}^d$,

$$-F(y, \check{v}(t,y)) \leq \frac{\check{v}(t,y)^2}{\eta(y)} + \theta\check{v}(t,y).$$

We estimate $\check{v}(t,y)^2$ using the inequality $(u-v)^2 \leq u^2 - uv$ for $u \geq v \geq 0$ and obtain,

$$-F(y, \check{v}(t,y)) \leq \frac{\eta(y) - \eta(y)\|\frac{\mathcal{L}\eta}{\eta}\|(T-t)}{e^{2\theta(T-t)}(T-t)^2} + \theta\check{v}(t,y). \tag{3.24}$$

Since $e^{-2\theta(T-t)} \leq e^{-\theta(T-t)}$, adding (3.23) and (3.24) yields

$$-\partial_t \check{v}(t,y) - \mathcal{L}\check{v}(t,y) - F(t, \check{v}(t,y)) \leq -\frac{\eta(y)\|\frac{\mathcal{L}\eta}{\eta}\| + \mathcal{L}\eta(y) - \mathcal{L}\eta(y)\|\frac{\mathcal{L}\eta}{\eta}\|(T-t)}{e^{\theta(T-t)}(T-t)}.$$

Using again that $1 \geq \|\frac{\mathcal{L}\eta}{\eta}\|(T-t)$ we obtain,

$$\begin{aligned}
&\eta(y)\|\frac{\mathcal{L}\eta}{\eta}\| + \mathcal{L}\eta(y) - \mathcal{L}\eta(y)\|\frac{\mathcal{L}\eta}{\eta}\|(T-t) \\
\geq &\eta(y)\|\frac{\mathcal{L}\eta}{\eta}\| \cdot \left[1 - \|\frac{\mathcal{L}\eta}{\eta}\|(T-t)\right] + \mathcal{L}\eta(y) - \mathcal{L}\eta(y)\|\frac{\mathcal{L}\eta}{\eta}\|(T-t) \\
= &\left[1 - \|\frac{\mathcal{L}\eta}{\eta}\|(T-t)\right] \cdot \left[\eta(y)\|\frac{\mathcal{L}\eta}{\eta}\| + \mathcal{L}\eta(y)\right] \geq 0.
\end{aligned}$$

Thus,

$$-\partial_t \check{v}(t,y) - \mathcal{L}\check{v}(t,y) - F(t, \check{v}(t,y)) \leq 0.$$

□

Proof of Theorem 3.1.7. From the definition of $\check{v}, \hat{v}$ we have

$$\begin{aligned}
&(T-t)\check{v}(t,y) = \eta(y) + \eta(y)O(T-t) \quad \text{uniformly in } y \text{ as } t \to T. \\
&(T-t)\hat{v}(t,y) = \eta(y) + (1+|y|^m)O(T-t) \quad \text{uniformly in } y \text{ as } t \to T.
\end{aligned} \tag{3.25}$$

Then for $\varepsilon = \frac{1}{2}$, there exists $\delta_0 \in (0, \delta]$ such that for all $t \in [T-\delta_0, T)$,

$$\check{v}(t,y)(T-t) > \eta(y) - \frac{1}{2}\eta(y) = \frac{1}{2}\eta(y) \quad \text{uniformly on } \mathbb{R}^d.$$

Since $\eta \in C_m(\mathbb{R}^d)$, we obtain from (3.25) that

$$\lim_{t\to T} \frac{\check{v}(t,y)(T-t) - \eta(y)}{1+|y|^m} = \lim_{t\to T} \frac{\hat{v}(t,y)(T-t) - \eta(y)}{1+|y|^m} = 0, \quad \text{uniformly on } \mathbb{R}^d. \tag{3.26}$$

In order to apply Perron's method, we set

$$\mathcal{S} = \{u | u \text{ is a subsolution of (3.13) on } [T-\delta_0, T) \times \mathbb{R}^d \text{ and } u \leq \hat{v}\}.$$

From Proposition 3.2.5 we know that $\check{v} \in \mathcal{S}$, so $\mathcal{S}$ is non-empty. Thus, the function

$$v(t,y) = \sup\{u(t,y) : u \in \mathcal{S}\}$$

is well-defined and belongs to $USC_m([T-\delta_0, T^-] \times \mathbb{R}^d)$. Classical arguments[3] show that the upper semi-continuous envelope v^* which equals v is a viscosity subsolution to (3.13). From [Zha99, Lemma A.2], the lower semi-continuous envelope v_* of v is also a viscosity supersolution to (3.13). Since $\check{v} \leq v_* \leq v^* \leq \hat{v}$, we have that for all $t \in [T-\delta_0, T)$,

$$v_*(t,y)(T-t), v^*(t,y)(T-t) \geq \frac{1}{2}\eta(y), \quad \text{uniformly on } \mathbb{R}^d.$$

[3] The standard Perron method of finding viscosity solutions for elliptic PDEs can be found in [CIL92]. We refer to [Zha99, Appendix A] for the proof of this method for parabolic equations.

and

$$\frac{\check{v}(t,y)(T-t)-\eta(y)}{1+|y|^m} \le \frac{v_*(t,y)(T-t)-\eta(y)}{1+|y|^m} \le \frac{v^*(t,y)(T-t)-\eta(y)}{1+|y|^m} \le \frac{\hat{v}(t,y)(T-t)-\eta(y)}{1+|y|^m}.$$

Hence, it follows from (3.26) that,

$$\lim_{t\to T}\frac{v_*(t,y)(T-t)-\eta(y)}{1+|y|^m} = \lim_{t\to T}\frac{v^*(t,y)(T-t)-\eta(y)}{1+|y|^m} = 0, \quad \text{uniformly on } \mathbb{R}^d. \tag{3.27}$$

From our comparison principle [Proposition 3.2.1] we conclude that $v^* = v \le v_*$ on $[T-\delta_0, T) \times \mathbb{R}^d$, which shows that v is the desired viscosity solution to (3.6) that belongs to $C_m([T-\delta_0, T^-] \times \mathbb{R}^d)$.

It follows from [AT96, Remark 6] that there exists a unique viscosity solution $v \in C_m([0, T-\delta_0] \times \mathbb{R}^d)$ to (3.6) when imposed at $t = T-\delta_0$ with a terminal value in $C_m(\mathbb{R}^d)$. Hence from the comparison principle for *continuous* viscosity solutions [GHS18, Lemma 3.1], we get a unique global viscosity solution

$$v \in C_m([0, T^-] \times \mathbb{R}^d).$$

□

Remark 3.2.6. If all the coefficients of the generator F and the SDE (3.10) are bounded, then one can show that twice differentiability of η is not needed; only a uniform continuity is required to choose continuous solutions which satisfying the conditions (3.16) and (3.17). Thus a unique viscosity solution can be obtained by the same argument above.

3.2.2. Verification

This section is devoted to the verification argument. Throughout, the function $v \in C_m([0,T^-] \times \mathbb{R}^d)$ denotes the unique nonnegative viscosity solution to the singular terminal value problem (3.13). We will prove that the viscosity solution is indeed the value function to our stochastic control problem.

Admissiblity

In a first step we are now going to show that the feedback control given in (3.14) is indeed admissible.

Lemma 3.2.7. *The pair of feedback controls* (ξ^*, μ^*) *given by* (3.14) *is admissible. The portfolio process* $(X_s^*)_{s\in[t,T]}$ *with respect to* (ξ^*, μ^*) *is monotone.*

Proof. Since $\check{v} \le v \le \hat{v}$ on $[T-\delta, T)$, we have for $r \in [T-\delta, T)$ that

$$\frac{1-\|\frac{\mathcal{L}\eta}{\eta}\|(T-r)}{e^{\theta(T-r)}(T-r)}\eta(Y_r^{t,y}) \le v(r, Y_r^{t,y}) \le \frac{1+\|\frac{\mathcal{L}\eta}{\eta}\|(T-r)}{T-r}\eta(Y_r^{t,y}) + \hat{h}(r, Y_r^{t,y}).$$

For $s \in [T-\delta, T)$,

$$
\begin{aligned}
|X_s^*| &\leq |x| \exp\left(-\int_t^s \frac{v(r, Y_r^{t,y})}{\eta(Y_r^{t,y})}\,dr \right) \\
&\leq |x| \exp\left(-\int_{T-\delta}^s \frac{1 - \|\frac{\mathcal{L}\eta}{\eta}\|(T-r)}{e^{\theta(T-r)}(T-r)}\,dr \right) \\
&\leq |x| \exp\left(\int_{T-\delta}^s \frac{e^{\theta(T-r)} - [1 - \|\frac{\mathcal{L}\eta}{\eta}\|(T-r)]}{e^{\theta(T-r)}(T-r)}\,dr \right) \exp\left(-\int_{T-\delta}^s \frac{1}{T-r}\,dr \right) \\
&\leq |x| \exp\left(\int_{T-\delta}^s \left[\frac{e^{\theta(T-r)} - 1}{e^{\theta(T-r)}(T-r)} + \frac{\|\frac{\mathcal{L}\eta}{\eta}\|}{e^{\theta(T-r)}} \right] dr \right) \cdot \frac{T-s}{\delta} \\
&\leq C|x| \frac{T-s}{\delta}.
\end{aligned}
\tag{3.28}
$$

The last inequality holds because $\lim\limits_{r \to T} \frac{e^{\theta(T-r)}-1}{e^{\theta(T-r)}(T-r)} = \theta$. As a result, $X_{T-}^* = 0$ and hence $X_T^* = 0$.

For controls ξ^*, μ^* given by (3.14), the process $(X_s^*)_{s \in [t,T]}$ is obviously monotone and μ is admissible. It remains to establish the integrability of ξ^*. Since $\hat{h}, 1/\eta, v$ are polynomial growth, we see that

$$
\begin{aligned}
\mathbb{E} \int_0^T |\xi_s^*|^4\,ds \leq& T \left(\mathbb{E}\left[\sup_{0 \leq s \leq T-\delta} |\xi_s^*|^4 + \sup_{T-\delta \leq s \leq T} |\xi_s^*|^4 \right] \right) \\
=& T \left(\mathbb{E}\left[\sup_{0 \leq s \leq T-\delta} \Big(\frac{v(s, Y_s^{t,y})}{\eta(Y_s^{t,y})} |X_s^*| \Big)^4 + \sup_{T-\delta \leq s \leq T} \Big(\frac{v(s, Y_s^{t,y})}{\eta(Y_s^{t,y})} |X_s^*| \Big)^4 \right] \right) \\
\leq& C|x|^4 \left(\mathbb{E}\left[\sup_{0 \leq s \leq T-\delta} \Big(\frac{v(s, Y_s^{t,y})}{\eta(Y_s^{t,y})} \Big)^4 \right] + \mathbb{E}\left[\sup_{T-\delta \leq s \leq T} \Big(\frac{\hat{h}(s, Y_s^{t,y})}{\eta(Y_s^{t,y})} \Big)^4 \right] \right) \\
&+ C|x|^4 \sup_{T-\delta \leq s \leq T} \Big(\frac{1 + \|\frac{\mathcal{L}\eta}{\eta}\|(T-s)}{T-s} \cdot \frac{T-s}{\delta} \Big)^4 \\
<& +\infty.
\end{aligned}
$$

It follows that $\xi^* \in L^4_{\mathcal{F}}(0,T;\mathbb{R})$ and hence that (ξ^*, μ^*) is admissible. ☐

Verification argument

It has been shown in [GHS18, Lemma 5.2] that we may w.l.o.g restrict ourselves to admissible controls that result in a monotone portfolio process. We denote by $\bar{\mathcal{A}}(t,x)$ the set of all admissible controls under which the portfolio process is monotone. For any $(\xi, \mu) \in \bar{\mathcal{A}}(t,x)$ the expected residual costs vanish as $s \to T$ as shown by the following lemma.

Lemma 3.2.8. *For every $(\xi, \mu) \in \bar{\mathcal{A}}(t,x)$ it holds that*

$$
\mathbb{E}\left[v(s, Y_s^{t,y}) |X_s^{\xi,\mu}|^2 \right] \longrightarrow 0, \quad s \to T. \tag{3.29}
$$

Proof. Following the proof in [GHS18, Lemma 5.3], we have

$$|X_s^{\xi,\mu}|^2 \leq C(T-s)E\left[\int_s^T |\xi_r|^2\,dr\Big|\mathcal{F}_s\right].$$

Therefore, recalling that $v \leq \hat{v}$,

$$\begin{aligned}
&\mathbb{E}\left[v(s,Y_s^{t,y})|X_s^{\xi,\mu}|^2\right]\\
\leq& C\mathbb{E}\left[\frac{\eta(Y_s^{t,y})+\hat{h}(s,Y_s^{t,y})(T-s)}{T-s}(T-s)E\Big[\int_s^T |\xi_r|^2\,dr\Big|\mathcal{F}_s\Big]\right]\\
\leq& C\sqrt{T-s}\cdot\mathbb{E}\left[\int_s^T |\xi_r|^4\,dr\right]^{1/2}\cdot\left(\mathbb{E}\left[\eta(Y_s^{t,y})^2\right]^{1/2}+\mathbb{E}\left[\hat{h}(s,Y_s^{t,y})^2\right]^{1/2}\right).
\end{aligned}$$

Letting $s\to T$, the desired result (3.29) is obtained by the fact that $\xi \in L^4_{\mathcal{F}}(0,T;\mathbb{R})$, $\eta \in C_m\left(\mathbb{R}^d\right)$ and $\hat{h} \in C_m\left([0,T]\times\mathbb{R}^d\right)$ along with Assumption 3.1.4 and the moment estimates (3.11) of Y. ☐

Next, we give a probabilistic representation of the viscosity solution to (3.13). In [Pop17], the author showed that the possibly discontinuous minimal solution of a certain backward stochastic differential equation with singular terminal condition gives a probabilistic representation of the minimal viscosity solution of an associated partial differential equation; continuity of the solution was *not* established. However, continuity is necessary to carry out the verification argument. We obtain a solution to the corresponding FBSDE in a different way since the existence of the (continuous) viscosity solution has already been proved.

Proposition 3.2.9. *Suppose that Assumptions 3.1.3 , 3.1.4 hold. There exists processes* $(U^{t,y},Z^{t,y}) \in S^2_{\mathcal{F}}(t,T^-;\mathbb{R})\times L^2_{\mathcal{F}}(t,T^-;\mathbb{R}^{1\times\tilde{d}})$ *satisfying that* $U_t^{t,y}=v(t,y)$ *and for any* $t\leq r\leq s<T$,

$$U_r^{t,y}=U_s^{t,y}+\int_r^s F(Y_\rho^{t,y},U_\rho^{t,y})d\rho-\int_t^s Z_\rho^{t,y}dW_\rho.$$

Proof. For fixed $T_0\in(0,T)$, we consider the forward-backward system

$$\begin{cases} dY_s=b(Y_s)ds+\sigma(Y_s)dW_s, & s\in[t,T_0],\\ dU_s=-f(s,Y_s)ds+Z_sdW_s, & s\in[t,T_0],\\ \quad Y_t=y, U_{T_0}=v(T_0,Y_{T_0}), \end{cases} \tag{3.30}$$

and the correponding PDE:

$$\begin{cases} -w_t(t,y)-\mathcal{L}w(t,y)-f(t,y)=0, & (t,y)\in[0,T_0)\times R^d,\\ w(T_0,y)=v(T_0,y), & y\in\mathbb{R}^d, \end{cases} \tag{3.31}$$

where $f(t,y):=F(y,v(t,y))$. Recalling the polynomial growth assumption on the coefficients in Assumption 3.1.4, we know that $f\in C_{m'}([0,T_0]\times\mathbb{R}^d)$ for some

$m' \geq m$. Together with Assumption 3.1.3 and the fact that $v(T_0, \cdot) \in C_{m'}(\mathbb{R}^d)$ derived by Theorem 3.1.7, we conclude from [KPQ97, Theorem 2.1] that the system admits a unique solution

$$(Y^{t,y}, U^{t,y}, Z^{t,y}) \in S^2_{\mathcal{F}}(t, T_0; \mathbb{R}^d) \times S^2_{\mathcal{F}}(t, T_0; \mathbb{R}) \times L^2_{\mathcal{F}}(t, T_0; \mathbb{R}^{1\times \tilde{d}}).$$

Let $w(t,y) := U^{t,y}_t$. By the Feynman-Kac formula, w is the unique viscosity solution of (3.31) with driver f. By the definition of f, we see that v is also a viscosity solution of (3.31) with driver f. Hence it follows that $w = v$. As a result, we have for any $r \in [t, T_0]$ that $0 \leq U^{t,y}_r = v(r, Y^{t,y}_r)$. Thus $U^{t,y}$ is also a solution to the following BSDE:

$$\begin{cases} dY_s = b(Y_s)ds + \sigma(Y_s)dW_s, & s \in [t, T_0], \\ dU_s = -F(Y_s, U_s)ds + Z_s dW_s, & s \in [t, T_0], \\ \ Y_t = y, U_{T_0} = v(T_0, Y_{T_0}). \end{cases} \tag{3.32}$$

We then can conclude that for any $T_0 < T$, the solution $U^{t,y}$ to the corresponding FBSDE system can always be expressed by the viscosity solution v. Therefore, a global solution to the BSDE on $[0, T)$ is obtained. □

The following lemma is key to the verification argument.

Lemma 3.2.10. *For every $(\xi, \mu) \in \bar{\mathcal{A}}(t, x)$ and $s \in [t, T)$,*

$$v(t,y)|x|^2 \leq E\left[v(s, Y^{t,y}_s)|X^{\xi,\mu}_s|^2\right] + E\left[\int_t^s c(Y^{t,y}_r, X^{\xi,\mu}_r, \xi_r, \mu_r)\, dr\right].$$

Proof. By Proposition 3.2.9, we know that $(U^{t,y}, Z^{t,y})$ solves the following BSDE:

$$U^{t,y}_t = U^{t,y}_s + \int_t^s F(Y^{t,y}_r, U^{t,y}_r)dr - \int_t^s Z^{t,y}_r dW_r.$$

This allows us to apply to $U^{t,y}_s |X^{\xi,\mu}_s|^2$ the classical integration by parts formula for semimartingales in order to obtain

$$\begin{aligned} U^{t,y}_t |x|^2 =& U^{t,y}_s |X^{\xi,\mu}_s|^2 + \int_t^s \big\{F(Y^{t,y}_r, U^{t,y}_r)|X^{\xi,\mu}_r|^2 \\ &+ 2\xi_r U^{t,y}_r \operatorname{sgn}(X^{\xi,\mu}_r)|X^{\xi,\mu}_r| - \theta U^{t,y}_r (|X^{t,x}_r - \mu_r|^2 - |X^{t,x}_r|^2)\big\}\, dr \\ &- \int_t^s \sigma(Y^{t,y}_r) Z^{t,y}_r |X^{\xi,\mu}_r|^2\, dW_r - \int_t^s U^{t,y}_r (|X^{\xi,\mu}_{r-} - \mu_r|^2 - |X^{\xi,\mu}_{r-}|^2)\, d\widetilde{N}_r, \end{aligned}$$

where $\widetilde{N}_r = N_r - \theta r$ denotes the compensated Poisson process. Moreover, due to the monotonicity of the portfolio process, $|X^{\xi,\mu}| \leq |x|$ and $|\mu| \leq |x|$. Furthermore,

$$\int_t^s \sigma(Y^{t,y}_r) Z^{t,y}_r |X^{\xi,\mu}_r|^2\, dW_r$$

is a uniformly integrable martingale because

$$
\begin{aligned}
&2\mathbb{E}\left[\left(\int_t^s |\sigma(Y_r^{t,y})|^2 \cdot |Z_r^{t,y}|^2 |X_r^{\xi,\mu}|^4 \, dr\right)^{1/2}\right] \\
\leq &\mathbb{E}\left(\sup_{t\leq r\leq s} |\sigma(Y_r^{t,y})|^2 + |x|^4 \int_t^s |Z_r^{t,y}|^2 \, dr\right) \\
<&\infty.
\end{aligned}
$$

As a consequence, the above stochastic integrals are true martingales. Hence, recalling (3.9),

$$
\begin{aligned}
U_t^{t,y}|x|^2 = &\, \mathbb{E}\left[U_s^{t,y}|X_s^{\xi,\mu}|^2\right] + \mathbb{E}\left[\int_t^s c(Y_r^{t,y}, X_r^{\xi,\mu}, \xi_r, \mu_r)\, dr\right] \\
&+ \mathbb{E}\left[\int_t^s \left\{F(Y_r^{t,y}, U_r^{t,y})|X_r^{\xi,\mu}|^2 - H(r, Y_r^{t,y}, X_r^{\xi,\mu}, \xi_r, \mu_r, U_r^{t,y}|X_r^{\xi,\mu}|^2)\right\} dr\right] \\
\leq &\, \mathbb{E}\left[U_s^{t,y}|X_s^{\xi,\mu}|^2\right] + \mathbb{E}\left[\int_t^s c(Y_r^{t,y}, X_r^{\xi,\mu}, \xi_r, \mu_r)\, dr\right]. \qquad (3.33)
\end{aligned}
$$

Since $U_t^{t,y} = v(t,y), U_r^{t,y} = v(r, Y_r^{t,y})$, we have

$$
v(t,y)|x|^2 \leq E\left[v(s, Y_s^{t,y})|X_s^{\xi,\mu}|^2\right] + E\left[\int_t^s c(Y_r^{t,y}, X_r^{\xi,\mu}, \xi_r, \mu_r)\, dr\right].
$$

□

We are now ready to carry out the verification argument.

Proof of Proposition 3.1.8. Let $(\xi, \mu) \in \bar{\mathcal{A}}(t,x)$. By Lemma 3.2.8 and Lemma 3.2.10 letting $s \to T$, we get

$$
v(t,y)|x|^2 \leq J(t,y,x;\xi,\mu).
$$

Finally, by Lemma 3.1.2 equality holds in (3.33) if $\xi = \xi^*$ and $\mu = \mu^*$. Hence, using Lemma 3.2.8, it yields

$$
\begin{aligned}
v(t,y)|x|^2 &= E[v(s, Y_s^{t,y})|X_s^{\xi^*,\mu^*}|^2] + E\left[\int_t^s c(Y_r^{t,y}, X_r^{\xi^*,\mu^*}, \xi_r^*, \mu_r^*)\, dr\right] \\
&\longrightarrow J(t,y,x;\xi^*,\mu^*) \quad \text{as } s \to T.
\end{aligned}
$$

This shows that the strategy (ξ^*, μ^*) is indeed optimal. □

3.3. Uniqueness in the non-Markovian framework

Throughout this section we assume that the filtration is solely generated by the Brownian motion. The existence of a minimal nonnegative solution

$$
(\mathcal{Y}, \mathcal{Z}) \in L^2_{\mathcal{F}}(\Omega; C([0,T^-]; \mathbb{R}_+)) \times L^2_{\mathcal{F}}(0,T^-; \mathbb{R}^{1\times \tilde{d}})
$$

to the BSDE

$$-d\mathcal{Y}_t = \left\{\lambda_t - \frac{|\mathcal{Y}_t|^2}{\eta_t}\right\} dt - \mathcal{Z}_t\, dW_t, \quad 0 \le t < T; \quad \lim_{t\to T} \mathcal{Y}_t = +\infty \tag{3.34}$$

has been established in [AJK14] under the assumption that $\eta \in L^2_{\mathcal{F}}(0,T;\mathbb{R}_+)$, $\eta^{-1} \in L^1_{\mathcal{F}}(0,T;\mathbb{R}_+)$, $\lambda \in L^2_{\mathcal{F}}(0,T^-;\mathbb{R}_+)$, and $\mathbb{E}[\int_0^T (T-t)^2 \lambda_t\, dt] < \infty$.

In this section we extend our uniqueness result to non-Markovian models and prove the existence of a unique nonnegative solution under the following conditions; they correspond to those in the Markovian setting..

Assumption 3.3.1. (i) The process η is a positive Itô diffusion satisfying that $d\eta_t = \alpha_t\, dt + \beta_t\, dW_t$ with $(\alpha,\beta) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}\times\mathbb{R}^{1\times\tilde{d}})$.

(ii) The processes $\eta, \eta^{-1} \in L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}))$ and $\eta^{-1}\alpha \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R})$.

(iii) There exists a positive Itô diffusion h_t such that $dh_t = \alpha'_t\, dt + \beta'_t\, dW_t$ with $(\alpha',\beta') \in L^2_{\mathcal{F}}(0,T;\mathbb{R}\times\mathbb{R}^{1\times\tilde{d}})$ and $h^{-1}\lambda, h^{-1}\alpha' \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R})$.

Proposition 3.3.2. *Let Asssumption 3.3.1 hold. Set $\tau := 1/\|\eta^{-1}\alpha\|_{L^\infty} \wedge T$ and $\tilde{K} := \|h^{-1}\alpha'\|_{L^\infty} + \|h^{-1}\lambda\|_{L^\infty}$. For any solution*

$$(\mathcal{Y},\mathcal{Z}) \in L^2_{\mathcal{F}}(\Omega; C([0,T^-];\mathbb{R}_+)) \times L^2_{\mathcal{F}}(0,T^-;\mathbb{R}^{1\times\tilde{d}})$$

to (3.34) *the following estimates hold for $T-\tau \le t < T$:*

$$\eta_t\left(\frac{1}{T-t} - \|\eta^{-1}\alpha\|_{L^\infty}\right) \le \mathcal{Y}_t \le \eta_t\left(\frac{1}{T-t} + \|\eta^{-1}\alpha\|_{L^\infty}\right) + e^{\tilde{K}(T-t)} h_t. \tag{3.35}$$

Proof. For $0 < \epsilon < \tau$ we define $(\overline{\mathcal{Y}_t}^\epsilon)_{t\in[T-\tau,T-\epsilon)}$ by

$$\overline{\mathcal{Y}_t}^\epsilon = \eta_t\left(\frac{1}{T-\epsilon-t} + \|\eta^{-1}\alpha\|_{L^\infty}\right) + e^{\tilde{K}(T-\epsilon-t)} h_t.$$

We will show that these processes are supersolutions to (3.34) but with the singularity at $t = T-\epsilon$,

$$\lim_{t\to T-\epsilon} \overline{\mathcal{Y}_t}^\epsilon = +\infty.$$

Precisely,

$$-d\overline{\mathcal{Y}_t}^\epsilon = g^\epsilon(t,\overline{\mathcal{Y}_t}^\epsilon)\, dt - \overline{\mathcal{Z}_t}^\epsilon\, dW_t, \qquad T-\tau \le t < T-\epsilon,$$

where

$$\begin{aligned} g^\epsilon(t,\overline{\mathcal{Y}_t}^\epsilon) := &- \frac{\eta_t}{(T-\epsilon-t)^2} - \alpha_t\left(\frac{1}{T-\epsilon-t} + \|\eta^{-1}\alpha\|_{L^\infty}\right) \\ &+ \tilde{K} e^{\tilde{K}(T-\epsilon-t)} h_t - e^{\tilde{K}(T-\epsilon-t)} \alpha'_t \end{aligned}$$

and $\overline{\mathcal{Z}}^\epsilon \in \bigcap_{t\in[T-\tau,T-\epsilon)} L^2_{\mathcal{F}}(T-\tau,t;\mathbb{R}^{1\times\tilde{d}})$. A calculation as in the proof of Proposition 3.2.5 verifies that for all $T-\tau \le t < T-\epsilon$,

$$g^\epsilon(t,\overline{\mathcal{Y}_t}^\epsilon) \ge \lambda_t - \frac{|\overline{\mathcal{Y}_t}^\epsilon|^2}{\eta_t} =: f(t,\overline{\mathcal{Y}_t}^\epsilon).$$

We now consider the difference of $\mathcal{Y}$ and $\overline{\mathcal{Y}}^\epsilon$ for $T-\tau \le t \le s < T-\epsilon$:

$$\begin{aligned}
\overline{\mathcal{Y}_t}^\epsilon - \mathcal{Y}_t &= \mathbb{E}\left[\overline{\mathcal{Y}_s}^\epsilon - \mathcal{Y}_s + \int_t^s g^\epsilon(r,\overline{\mathcal{Y}_r}^\epsilon)\,dr - \int_t^s f(r,\mathcal{Y}_r)\,dr\Big|\mathcal{F}_t\right] \\
&\ge \mathbb{E}\left[\overline{\mathcal{Y}_s}^\epsilon - \mathcal{Y}_s + \int_t^s f(r,\overline{\mathcal{Y}_r}^\epsilon) - f(r,\mathcal{Y}_r)\,dr\Big|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\overline{\mathcal{Y}_s}^\epsilon - \mathcal{Y}_s + \int_t^s (\overline{\mathcal{Y}_r}^\epsilon - \mathcal{Y}_r)\Delta_r\,dr\Big|\mathcal{F}_t\right]
\end{aligned}$$

where

$$\Delta_r = \begin{cases} \dfrac{f(r,\overline{\mathcal{Y}_r}^\epsilon) - f(r,\mathcal{Y}_r)}{\overline{\mathcal{Y}_r}^\epsilon - \mathcal{Y}_r}, & \textit{if} \quad \overline{\mathcal{Y}_r}^\epsilon - \mathcal{Y}_r \ne 0, \\ 0, & \textit{else}. \end{cases}$$

Note that $\Delta \le 0$. By the explicit representation of the solution to linear BSDEs,

$$\overline{\mathcal{Y}_t}^\epsilon - \mathcal{Y}_t \ge \mathbb{E}\left[(\overline{\mathcal{Y}_s}^\epsilon - \sup_{t\le s\le T-\epsilon} \mathcal{Y}_s)\exp\left(\int_t^s \Delta_r\,dr\right)\right].$$

Since $\overline{\mathcal{Y}_s}^\epsilon \ge 0$, $\mathbb{E}[\sup_{t\le s\le T-\epsilon}\mathcal{Y}_s] < +\infty$ due to $\mathcal{Y} \in L^2_{\mathcal{F}}(\Omega;C([0,T^-];\mathbb{R}_+))$, we can apply Fatou's lemma to the expectation above as $s \to T-\epsilon$ to obtain that $\overline{\mathcal{Y}_t}^\epsilon - \mathcal{Y}_t \ge 0$. Taking $\epsilon \to 0$ we obtain the upper estimate. The lower estimate can be established by similar arguments. □

Lemma 3.3.3. *Suppose that Asssumption 3.3.1 holds. Let $(\mathcal{Y},\mathcal{Z})$ be a solution of (3.34) in the space $L^2_{\mathcal{F}}(\Omega;C([0,T^-];\mathbb{R}_+)) \times L^2_{\mathcal{F}}(0,T^-;\mathbb{R}^{1\times\tilde{d}})$. Denote the associated portfolio process by*

$$X_t^* = \exp\left(-\int_0^t \frac{\mathcal{Y}_s}{\eta_s}\,ds\right).$$

Then we have that $X^*\mathcal{Z} \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$.

Proof. Let $M_t = \mathcal{Y}_t X_t^* + \int_0^t \lambda_s X_s^*\,ds$. Integration by parts yields

$$dM_t = X_t^* \mathcal{Z}_t dW_t. \tag{3.36}$$

Hence, M is a nonnegative local martingale on $[0,T)$ and in particular a nonnegative supermartingale. Thus, it converges almost surely in $\mathbb{R}$ as t goes to T. Similarly to (3.28), we use the lower estimate in (3.35) to obtain that for $s \in [T-\tau,T)$

$$|X_s^*| \le C(T-s).$$

In view of the upper estimate in (3.35), we have that

$$\mathbb{E}\left[\sup_{T-\tau\le t\le s}|\mathcal{Y}_t X_t^*|^2\right]\le C\mathbb{E}\left[\sup_{T-\tau\le t\le T}(|\eta_t|^2+|h_t|^2)\right],$$

where the constant C is independent of s. Thus, applying the dominated convergence theorem implies

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T}|M_t|^2\right]\\ \le& C\left(\mathbb{E}\left[\sup_{0\le t\le T-\tau}|\mathcal{Y}_t|^2\right]+\mathbb{E}\left[\sup_{T-\tau\le t\le T}(|\eta_t|^2+|h_t|^2)\right]+\mathbb{E}\left[\int_0^T|\lambda_s|^2\,ds\right]\right)\\ <&+\infty.\end{aligned}$$

Recalling the equation (3.36), we have that $X^*\mathcal{Z}\in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ and that M is indeed a nonnegative martingale on $[0,T]$. $\square$

It follows from [AJK14, Proposition 3.4] that $\mathcal{Y}$ is the minimal solution of (3.34). Therefore, we can obtain the uniqueness result.

Theorem 3.3.4. *Under Assumption 3.3.1, there exists a unique solution to the BSDE* (3.34) *in* $L^2_{\mathcal{F}}(\Omega;C([0,T^-];\mathbb{R}_+))\times L^2_{\mathcal{F}}(0,T^-;\mathbb{R}^{1\times\tilde{d}})$.

4. Portfolio liquidation under factor uncertainty

In this Chapter, we study a class of Markovian single-player portfolio liquidation problems where the investor is uncertain about the factor dynamics driving trading costs. The benchmark case has been analyzed in Chapter 3 with dark pools. We describe the modelling set-up, introduce the stochastic control problem and state our main results in Section 4.1. The existence of a unique viscosity solution to the HJBI equation is established in Section 4.2; the regularity of the viscosity solution is proved in Section 4.3. The verification argument is carried out in Section 4.4. Finally, Section 4.5 is devoted to an asymptotic analysis of the value function for small amounts of uncertainty.

4.1. Problem formulation and main results

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space that satisfies the usual conditions and carries an $\tilde{d}$-dimensional standard Brownian motion W and an independent one-diemensional standard Brownian motion B.

In this section we consider the problem of a large investor that needs to liquidate a given portfolio $x \in \mathbb{R}$ within the time horizon $[0, T]$. Let $t \in [0, T)$ be a given point in time and $x \in \mathbb{R}$ be the portfolio position of the trader at time t. We denote by $\xi_s \in \mathbb{R}$ the rate at which the agent trades at time $s \in [t, T)$. Given a trading strategy ξ, the portfolio position at time $s \in [t, T)$ is given by

$$X_s = x - \int_t^s \xi_r \, dr, \quad s \in [t, T]$$

and the liquidation constraint is

$$X_T = 0. \tag{4.1}$$

In what follows we assume that all trading costs are driven by a factor process given by the d-dimensional Itô diffusion

$$\begin{cases} dY_s^{t,y} = b(Y_s^{t,y})ds + \sigma(Y_s^{t,y})dW_s, & s \in [t, T], \\ Y_t^{t,y} = y. \end{cases}$$

Our goal is to analyze the impact of uncertainty about the factor dynamics on optimal liquidation strategies and trading costs.

4.1.1. The benchmark model

In this section we briefly recall the liquidation model without factor uncertainty analyzed in Chapter 3 and Graewe et al. [GHS18] against which our results shalll be benchmarked. We assume that the investor's transaction price $P_s \in \mathbb{R}$ at time $s \in [t,T]$ can additively decomposed into a fundamental asset price $\tilde{P}_s$ and an instantaneous price impact term $f(\xi_s)$ as

$$P_s = \tilde{P}_s - f(\xi_s)$$

where the fundamental asset price process $\tilde{P}$ is given by a one-dimensional square-integrable Brownian martingale, which we assume to be of the form[1]

$$d\tilde{P}_s = \tilde{\sigma}(Y_s^{t,y})dB_s$$

for some function $\tilde{\sigma}$. The investor aims at minimizing the difference between the book value of the portfolio and the expected proceeds from trading plus risk cost. Assume that the instantaneous price impact factor $f(\xi_s) := \eta(Y_s^{t,y})|\xi_s|^{p-1}\operatorname{sgn}(\xi_s)$ for some $p > 1$ and some positive function η that describes the inverse market depth and that the risk is measured by the integral of the pth power of the value at risk of an open position over the trading period. Assume that any admissible trading strategy ξ belongs to $L^{2p}_{\mathcal{F}}(t,T;\mathbb{R})$. The resulting cost functional is then given by

$$\begin{aligned} J(t,y,x,\xi) &= \text{book value} - \text{expected proceeds from trading} + \text{risk costs} \\ &= \mathbb{E}_{\mathbb{P}}\Big[\int_t^T \eta(Y_s^{t,y})|\xi_s|^p ds + \int_t^T X_s d\tilde{P}_s + \int_t^T \lambda(Y_s^{t,y})|X_s|^p\, ds\Big] \\ &= \mathbb{E}_{\mathbb{P}}\Big[\int_t^T \big(\eta(Y_s^{t,y})|\xi_s|^p + \lambda(Y_s^{t,y})|X_s|^p\big)\, ds\Big], \end{aligned} \tag{4.2}$$

where the last equality follows from the facts that $X \in \mathcal{S}^2_{\mathcal{F}}(\Omega; C([t,T];\mathbb{R}))$ and that $\tilde{P}$ is a square-integrable martingale under $\mathbb{P}$.

For each initial state $(t,y,x) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}$ the value function of the investor's control problem is defined by

$$V_0(t,y,x) := \inf_{\xi \in \mathcal{A}(t,x)} J(t,y,x,\xi) \tag{4.3}$$

where the infimum is taken over the set $\mathcal{A}(t,x)$ of all admissible controls, that is, over all the controls ξ that belong to $L^{2p}_{\mathcal{F}}(t,T;\mathbb{R})$ and that satisfy the liquidation constraint (4.1). Under suitable assumptions on the model parameters it was shown in Chapter 3 that the value function is given by $V_0 = v_0|x|^p$ and that the optimal trading strategy is given by $\xi_0^*(t,y,x) = \frac{v_0(t,y)^\beta}{\eta(y)^\beta}x$ where $\beta = \frac{1}{p-1}$ and where v_0 is the unique nonnegative viscosity solution of polynomial growth to the following PDE:

$$\begin{cases} -\partial_t v(t,y) - \mathcal{L}v(t,y) - F(y,v(t,y)) = 0, & (t,y) \in [0,T) \times \mathbb{R}^d, \\ \lim_{t\to T} v(t,y) = +\infty, & \text{locally uniformly on } \mathbb{R}^d, \end{cases} \tag{4.4}$$

where

$$F(y,v) := \lambda(y) - \frac{|v|^{\beta+1}}{\beta\eta(y)^\beta}.$$

[1] See Example 4.1.3 below for a stochastic volatility model with uncertainty about the driver of the volatility process.

4.1.2. The liquidation model under uncertainty

In order to analyze the impact of factor uncertainty on optimal liquidation strategies we introduce the class $\mathcal{Q}$ of all probability measures Q whose density with respect to the benchmark measure $\mathbb{P}$ is given by

$$\frac{dQ}{d\mathbb{P}} = \mathcal{E}\left(\int_t \vartheta_s dW_s\right)_T, \quad Q\text{-}a.s.$$

for some progressively measurable process ϑ satisfying that

$$\int_t^T |\vartheta_s|^2 ds < \infty, \quad Q\text{-}a.s..$$

Here, $\mathcal{E}(M)_T = \exp(M_T - \frac{\langle M \rangle_T}{2})$ denotes the Doleans-Dade exponential of a continuous semimartingale M.

Since our focus is on the impact of uncertainty about the factor dynamics on the optimal trading rules, we assume that the Brownian motions B and W are independent. In this case the unaffected price process is still a square-integrable martingale under every probability $Q \in \mathcal{Q}$. In view of (4.2), we thus obtain the same form for the cost function for every given probability Q in the set $\mathcal{Q}$:

$$J_Q(t,y,x,\xi) = \mathbb{E}_Q\Big[\int_t^T \left(\eta(Y_s^{t,y})|\xi_s|^p + \lambda(Y_s^{t,y})|X_s|^p\right) ds\Big].$$

Following a standard approach in optimal decision making under model uncertainty introduced by Hansen and Sargent [HS01], we do not restrict the set of measures *a priori* but add a penalty term to the objective function. Specifically, every probability measure $Q \in \mathcal{Q}$ receives a penalty

$$\Upsilon(Q) := \mathbb{E}_Q\left[\int_t^T \frac{1}{\hat{\theta}_s}|\vartheta_s|^m ds\right].$$

The nonnegative process $\hat{\theta} = (\hat{\theta}_s)$ measures the degree of confidence in the reference model: the larger the process, the less deviations from the reference model are penalized. The case $\hat{\theta}_s \equiv 0$ corresponds to the benchmark model without factor uncertainty. The case $\hat{\theta}_s \equiv \hat{\theta}$ and $m = 2$ corresponds to the entropic penalty function, see, e.g. [AHS03, BMS07].

To the best of our knowledge, Maenhout [Mae04] was the first to propose a state-dependent parameter $\hat{\theta}$ when considering the robust portfolio optimization problem of a power-utility investor. He considered an uncertainty-tolerance parameter of the $\hat{\theta}_s = \frac{\theta}{\mathcal{W}_s^{1-r}}$ where θ is a positive constant, $\mathcal{W}_s$ denotes the wealth of the investor at time s and $r \in (0,1)$ denotes the exponent in the power utility function. This choice of $\hat{\theta}$ essentially corresponds to scaling the uncertainty-tolerance parameter by the value function. In his model, this leads to a solution that is invariant to the scale of wealth and is amenable to a rigorous mathematical analysis. Among other things, he found that for this choice of homothetic preferences the optimal solution under model uncertainty is observationally equivalent to the optimal solution without model uncertainty but increased risk aversion.

In our context, the approach of Maenhout [Mae04] corresponds to the choice

$$\hat{\theta}_s := \frac{\theta}{a|X_s^\xi|^p}$$

and thus to the penalty functional

$$\Upsilon(Q) := \mathbb{E}_Q\left[\int_t^T \frac{1}{\theta} a|\vartheta_s|^m |X_s^\xi|^p ds\right],$$

where the constant $a := \frac{(m-1)^{m-1}}{m^m}$ is chosen for analytical convenience. We thus model the costs associated with an admissible trading strategy ξ and probability measure $Q \in \mathcal{Q}$ by

$$\tilde{J}(t,y,x;\xi,\vartheta) := \mathbb{E}_Q\left[\int_t^T \left(\eta(Y_s^{t,y})|\xi_s|^p + \lambda(Y_s^{t,y})|X_s^\xi|^p - \frac{1}{\theta}a|\vartheta_s|^m|X_s^\xi|^p\right) ds\right]$$

define the value function of the stochastic control problem for each initial state $(t,y,x) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}$ as

$$V(t,y,x) := \inf_{\xi \in \mathcal{A}(t,x)} \sup_{Q \in \mathcal{Q}} \tilde{J}(t,y,x;\xi,\vartheta). \tag{4.5}$$

We assume throughout that $p > 1, m \geq 2, \theta \leq 1$. Before presenting the main results, we list our assumptions on the model parameters in terms of some positive constants $\underline{c}, \bar{C}$.

Assumption 4.1.1. (on the diffusion coefficients)

(L.1) The drift function $b : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous and of linear growth, i.e. for each $y \in \mathbb{R}^d$,

$$|b(x) - b(y)| \leq \bar{C}|x-y|, \quad |b(y)| \leq \bar{C}(1+|y|).$$

(L.2) The volatility function $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times\tilde{d}}$ is Lipschitz continuous and of linear growth, i.e. for each $y \in \mathbb{R}^d$,

$$|\sigma(x) - \sigma(y)| \leq \bar{C}|x-y|, \quad |\sigma(y)| \leq \bar{C}(1+|y|).$$

(L.3) The volatility function σ is uniformly bounded by $\bar{C}$.

(L.4) The drift and volatility functions b, σ belong to C^1 and $\sigma\sigma^*$ is uniformly positive definite.

Assumption 4.1.2. (on the cost coefficients)

(F.1) The coefficients $\eta, \lambda, 1/\eta : \mathbb{R}^d \to [0, \infty)$ are continuous. Moreover, there exists constants $k_0 \in (0, 1]$ such that for $y \in \mathbb{R}^d$,

$$\lambda(y) \leq \bar{C}\langle y\rangle^{(1-k_0)m}$$

and

$$\underline{c}\,\langle y\rangle^{(1-pk_0)m} \leq \eta(y) \leq \bar{C}\langle y\rangle^{(1-k_0)m}.$$

Let $\tilde{m} := (1 - k_0)m$.

(F.2) The function η is twice continuously differentiable and $\left\|\frac{\mathcal{L}\eta}{\eta}\right\|, \left\|\frac{|D\eta|^{\alpha+1}}{\eta}\right\| \leq \bar{C}$ where

$$\mathcal{L} := \frac{1}{2}\mathrm{tr}(\sigma\sigma^* D^2) + \langle b, D\rangle\,, \quad \alpha := \frac{1}{m-1}.$$

(F.3) The function λ belongs to $C_b^1(\mathbb{R}^d)$ and $0 < \underline{c} \leq \eta \leq \bar{C}$.

The assumptions on the diffusion coefficients are standard. Assumption (F.1) states that λ is of polynomoial growth and that η can be bounded from below and above by polynomial growth functions, whose order may be negative. Conditions similar to (F.2) and (F.3) have also been made in Chapter 3 and [GHS18], respectively.

Example 4.1.3. The assumptions on the diffusion coefficients are satisfied for the two-dimensional diffusion process $Y = (Y^1, Y^2)$ given by

$$dY_t^1 = -Y_t^1 dt + dW_t^1 \quad \text{and} \quad dY_t^2 = \mu dt + \sigma dW_t^2.$$

The Ornstein-Uhlenbeck process Y^1 drives the market impact term while the arithmetic Brownian motion Y^2 drives the market risk. Specifically, if we chose $\eta = \tanh(-Y^1) + 2$, then this process can be viewed as describing a stochastic liquidity process that fluctuates around a stationary level. Moreover, for the stochastic volatility model

$$d\tilde{P}_t = \tilde{\sigma}(Y_t^2)dB_t$$

for the reference price process the instantaneous volatility of the portfolio process is given by $\tilde{\sigma}^2(Y_t^2)|X_t|^2$. Hence, if $\tilde{\sigma}$ is bounded and continuously differentiable with bounded derivative, then $\lambda := \tilde{\sigma}^2$ satisfies the preceding assumptions.

4.1.3. The main results

If all the processes ϑ take values in a compact set Θ then all probability measures Q in $\mathcal{Q}$ are equivalent to $\mathbb{P}$. In this case, the dynamic programming principle suggests that the value function satisfies the following Hamilton-Jacobi-Bellman-Isaacs equation, cf. [FS89, Theorem 2.6]

$$-\partial_t V(t,y,x) - \mathcal{L}V(t,y,x) - \inf_{\xi\in\mathbb{R}} \sup_{\vartheta\in\Theta} \mathcal{H}(t,y,x,\xi,\vartheta,V) = 0, (t,y,x) \in [0,T) \times \mathbb{R}^d \times \mathbb{R},$$

where $\mathcal{H}$ is given by

$$\mathcal{H}(t,y,x,\xi,\vartheta,V) := \langle \sigma\vartheta, \partial_y V(t,y,x)\rangle - \xi\partial_x V(t,y,x) + c(y,x,\xi) - \frac{1}{\theta}a|\vartheta|^m|x|^p,$$

and

$$c(y,x,\xi) := \eta(y)|\xi|^p + \lambda(y)|x|^p.$$

In our case the set of probability measures is not restricted *a priori*. This suggests to characterize the value function (4.5) in terms of the solution to the modified HJBI equation

$$-\partial_t V(t,y,x) - \mathcal{L}V(t,y,x) - \inf_{\xi\in\mathbb{R}} \sup_{\vartheta\in\mathbb{R}^{\bar{d}}} \mathcal{H}(t,y,x,\xi,\vartheta,V) = 0, (t,y,x) \in [0,T)\times\mathbb{R}^d\times\mathbb{R}.$$

Since the function $\mathcal{H}$ separates additively into two terms that depend on ϑ only and into two terms that depend ξ only,

$$\begin{aligned}\inf_{\xi\in\mathbb{R}} \sup_{\vartheta\in\mathbb{R}^{\bar{d}}} \mathcal{H}(t,y,x,\xi,\vartheta,V) = &\sup_{\vartheta\in\mathbb{R}^{\bar{d}}}\{\langle \sigma\vartheta, \partial_y V(t,y,x)\rangle - \frac{1}{\theta}a|\vartheta|^m|x|^p\}\\ &+ \inf_{\xi\in\mathbb{R}}\{-\xi\partial_x V(t,y,x) + c(y,x,\xi)\}.\end{aligned}$$

The structure of cost function suggests an ansatz of the form $V(t,y,x) = v(t,y)|x|^p$. In this case,

$$\begin{aligned}\vartheta^*(t,y) :=& \arg\max_{\vartheta\in\mathbb{R}^{\bar{d}}} \left\{\left\langle \sigma\vartheta, Dv(t,y)\right\rangle - \frac{1}{\theta}a|\vartheta|^m\right\}\\ =&\theta^\alpha(1+\alpha)|\sigma^*(y)Dv(t,y)|^{\alpha-1}\sigma^*(y)Dv(t,y),\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\xi^*(t,y) :=& \arg\min_{\xi\in\mathbb{R}} \left\{ - p\xi v(t,y)|x|^{p-1}\operatorname{sgn}(x) + \eta(y)|\xi|^p\right\}\\ =&\frac{v(t,y)^\beta}{\eta(y)^\beta}x,\end{aligned}$$

where $\alpha = \frac{1}{m-1}, \beta = \frac{1}{p-1}$. Thus,

$$\inf_{\xi\in\mathbb{R}} \sup_{\vartheta\in\mathbb{R}^{\bar{d}}} \mathcal{H}(t,y,x,\xi,\vartheta,V) = \Big(H(y,Dv(t,y)) + F(y,v(t,y))\Big)|x|^p$$

where

$$F(y,v) := \lambda(y) - \frac{|v|^{\beta+1}}{\beta\eta(y)^\beta}, \quad H(y,q) := \theta^\alpha|\sigma^*(y)q|^{\alpha+1}. \tag{4.7}$$

Similarly to the discussion in [GHS18, Section 2.2], we expect the value function to be characterized by the following terminal value problem:

$$\begin{cases} -\partial_t v(t,y) - \mathcal{L}v(t,y) - H(y,Dv(t,y)) - F(y,v(t,y)) = 0, & (t,y)\in[0,T)\times\mathbb{R}^d,\\ \lim_{t\to T} v(t,y) = +\infty, & \text{locally uniformly on } \mathbb{R}^d.\end{cases} \tag{4.8}$$

The problem reduces to the terminal value problem (4.4) in the absence of model uncertainty ($H = 0$). The following theorem guarantees the existence of a unique nonnegative viscosity solution to this singular problem under conditions (L.1)-(L.3), (F.1), (F.2) and $\beta > \alpha$. The additional assumption $\beta > \alpha$ can also be found in [GKLV16] where the authors study the entire solutions of a similar kind of elliptic equation. The proof is given in Section 4.2.

Theorem 4.1.4. *Let $\beta > \alpha$. Under Assumptions (L.1)-(L.3), (F.1) and (F.2), the singular terminal value problem* (4.8) *admits a unique nonnegative viscosity solution v in*

$$C_{\tilde{m}}([0, T^-] \times \mathbb{R}^d),$$

where $\tilde{m}$ is introduced in condition (F.1).

Since the maximizer ϑ^* in (4.6) depends on Dv, we expect the verification theorem to require the candidate value function v to be of class $C^{0,1}$. As shown by the following theorem this can be guaranteed under additional assumptions on the model parameters. Specifically, we show that uniformly in y as $t \to T$ the function v satisfies

$$(T-t)^{1/\beta} v(t, y) = \eta(y) + O((T-t)^{1-\alpha/\beta}),$$

and

$$(T-t)^{1/\beta} Dv(t, y) = D\eta(y) + O((T-t)^{\frac{1}{2}-\alpha/\beta}).$$

Thus, under the additional assumption that $\beta > 2\alpha$, we obtain the convergence of both the rescaled function v and its rescaled derivative to market impact term, respectively its derivative at the terminal time:

$$\lim_{t \to T} (T-t)^{1/\beta} v(t, y) = \eta(y), \quad \lim_{t \to T} (T-t)^{1/\beta} Dv(t, y) = D\eta(y).$$

The proof of the following theorem is given in Section 4.3.

Theorem 4.1.5. *Let $\beta > 2\alpha$. Under Assumptions (L.1)-(L.4), (F.1)-(F.3), the unique nonnegative viscosity solution v in $C_b([0, T^-] \times \mathbb{R}^d)$ to the singular terminal value problem* (4.8) *belongs to $C^{0,1}([0, T) \times \mathbb{R}^d)$.*

The previously established regularity of the candidate value function is indeed enough to carry out the verification argument, which is proven in Section 4.4.

Theorem 4.1.6. *Let $\beta > 2\alpha$. Under Assumptions (L.1)-(L.4), (F.1)-(F.3), let $v \in C^{0,1}([0, T) \times \mathbb{R}^d)$ be the nonnegative viscosity solution to the singular terminal value problem* (4.8)*. Then, the value function of the control problem* (4.5) *is given by $V(t, y, x) = v(t, y)|x|^p$, and the optimal control (ξ^*, ϑ^*) is given in feedback form by*

$$\begin{cases} \xi_s^* = \dfrac{v(s, Y_s^{t,y})^\beta}{\eta(Y_s^{t,y})^\beta} X_s^*, \\ \vartheta_s^* = \theta^\alpha (1+\alpha) |\sigma^*(Y_s^{t,y}) Dv(s, Y_s^{t,y})|^{\alpha-1} \sigma^*(Y_s^{t,y}) Dv(s, Y_s^{t,y}). \end{cases} \tag{4.9}$$

In particular, the resulting optimal portfolio process $(X_s^*)_{s\in[t,T]}$ *is given by*

$$X_s^* = x\exp\left(-\int_t^s \frac{v(r,Y_r^{t,y})^\beta}{\eta(Y_r^{t,y})^\beta}\,dr\right). \tag{4.10}$$

Remark 4.1.7. The preceding results shows that – as in [Mae04] – the model with factor uncertainty is equivalent to the benchmark model (4.2) when the market risk factor λ is replaced by $\lambda^H := \lambda + H(y, Dv(t,y))$. In particular, under model uncertainty the investor liquidates the asset at a faster rate.

Ou final results provides a first order approximation of the value for the model with uncertainty in terms of the solution to the benchmark model without uncertainty when the investor is “almost certain” about the reference model.

Theorem 4.1.8. *Let* $\beta > 2\alpha$. *Let* $w = v(T-t)^{1/\beta}$ *and* $w_0 = v_0(T-t)^{1/\beta}$ *where* v_0 *denotes the solution to the benchmark model. Under Assumptions (L.1)-(L.4), (F.1)-(F.3), we have that on* $[0,T]\times\mathbb{R}^d$,

$$\lim_{\theta\to 0}\frac{w-w_0}{\theta^\alpha} = w_1 \tag{4.11}$$

where, w_1 *is a unique nonnegative solution to the following PDE:*

$$\begin{cases} -\partial_t v(t,y) - \mathcal{L}v(t,y) - f_1(t,y,v(t,y)) = 0, & (t,y)\in[0,T)\times\mathbb{R}, \\ v(T,y) = 0, & y\in\mathbb{R}^d. \end{cases} \tag{4.12}$$

whose driver

$$f_1(t,y,v) = |\sigma^* Dv_0|^{1+\alpha}(T-t)^{1/\beta} - \frac{(\beta+1)v_0^\beta}{\beta\eta^\beta}v + \frac{1}{\beta}\frac{v}{(T-t)}$$

depends on the solution to the benchmark model without factor uncertainty.

4.2. Viscosity solution

In this section, we prove Theorem 4.1.4. The proof uses modifications of arguments given in Chapter 3. We start with the following comparison principle. We emphasize that the comparison principle will only be used to prove the existence of a viscosity solution. This justifies the rather strong assumptions (4.13) and (4.14) below.

Proposition 4.2.1. *Assume that Assumptions (L.1)-(L.3), (F.1) and (F.2) hold. Let* $\tilde{m}$ *be as in condition (F.1). Fix* $\delta\in(0,T]$. *Let* $\overline{u}\in LSC_{\tilde{m}}([T-\delta,T^-]\times\mathbb{R}^d)$ *and* $\underline{u}\in USC_{\tilde{m}}([T-\delta,T^-]\times\mathbb{R}^d)$ *be a nonnegative viscosity super- and a viscosity subsolution to* (4.8)*, respectively. If, uniformly on* $\mathbb{R}^d$,

$$\limsup_{t\to T}\frac{\underline{u}(t,y)(T-t)^{1/\beta}-\eta(y)}{\langle y\rangle^{\tilde{m}}} \le 0 \le \liminf_{t\to T}\frac{\overline{u}(t,y)(T-t)^{1/\beta}-\eta(y)}{\langle y\rangle^{\tilde{m}}}, \tag{4.13}$$

and

$$\sqrt[\beta]{\frac{\frac{1}{2}\beta+1}{\beta+1}}\eta(y) \leq \underline{u}(t,y)(T-t)^{1/\beta}, \quad \overline{u}(t,y)(T-t)^{1/\beta} \leq C\langle y\rangle^{\tilde{m}}, \quad t \in [T-\delta, T), \tag{4.14}$$

for a constant C, *then*

$$\underline{u} \leq \overline{u} \quad on \quad [T-\delta, T) \times \mathbb{R}^d.$$

We first introduce three auxiliary results. Under assumptions (F.1), (F.2) and (4.14), the maps $(t,y) \mapsto (T-t)^{1/\beta}\underline{u}(t,y), (T-t)^{1/\beta}\overline{u}(t,y)$ satisfy the condition (A.3) in Proposition A.3.2. Let us fix

$$\rho \in \left(\sqrt[\beta]{\frac{\frac{1}{4}\beta+1}{\frac{1}{2}\beta+1}}, 1\right)$$

and consider the difference

$$w := \underline{u} - \rho\overline{u} \in USC_{\tilde{m}}([T-\delta, T^-] \times \mathbb{R}^d) \subset \mathcal{SSG}^-_m([T-\delta, T^-] \times \mathbb{R}^d).$$

The proof of the following lemma is similar to that of Proposition A.3.2.

Lemma 4.2.2. *The function* w *is a viscosity subsolution to*

$$\begin{aligned} &-\partial_t w(t,y) - \mathcal{L}w(t,y) - (\frac{1-\rho}{2})^{-\alpha}\bar{C}^{\alpha+1}|Dw|^{\alpha+1} - l(t,y)w(t,y) \\ &-(1-\rho)\left[\lambda(y) + \frac{1+\beta}{\beta}\frac{\hat{C}\langle y\rangle^m}{(T-t)^{1/\beta+1}}\right] = 0, \quad (t,y) \in [T-\delta, T) \times \mathbb{R}^d \end{aligned} \tag{4.15}$$

where

$$l(t,y) := \frac{F(y, \underline{u}(t,y)) - F(y, \rho\overline{u}(t,y))}{\underline{u}(t,y) - \rho\overline{u}(t,y)}\mathbb{I}_{\underline{u}(t,y)\neq\rho\overline{u}(t,y)}.$$

The next lemma constructs a local smooth strict supersolution to (4.15).

Lemma 4.2.3. *There exists* $L, C, \tau > 0$ *such that*

$$\chi(t,y) := (1-\rho)\frac{e^{L(T-t)}C\langle y\rangle^m}{(T-t)^{1/\beta}}$$

satisfies

$$\begin{aligned} \mathcal{J}[\chi] := &-\partial_t\chi(t,y) - \mathcal{L}\chi(t,y) - (\frac{1-\rho}{2})^{-\alpha}\bar{C}^{\alpha+1}|D\chi(t,y)|^{\alpha+1} + \frac{1+\frac{1}{4}\beta}{\beta(T-t)}\chi(t,y) \\ &-(1-\rho)\left[\lambda(y) + \frac{1+\beta}{\beta}\frac{\hat{C}\langle y\rangle^m}{(T-t)^{1/\beta+1}}\right] > 0, \quad (t,y) \in [T-\tau, T) \times \mathbb{R}^d. \end{aligned} \tag{4.16}$$

Proof. Set $\psi(t,y) := (1-\rho)e^{L(T-t)}C\langle y\rangle^m$. Analogous to the proof of Proposition A.3.2, we have

$$\mathcal{L}\chi(t,y) \le [2m\bar{C} + 2m(m-1)\bar{C}^2]\frac{\psi(t,y)}{(T-t)^{1/\beta}},$$
$$(\frac{1-\rho}{2})^{-\alpha}\bar{C}^{\alpha+1}|D\chi(t,y)|^{\alpha+1} \le [2^\alpha m^{\alpha+1}\bar{C}^{\alpha+1}C^\alpha e^{\alpha L(T-t)}]\frac{\psi(t,y)}{(T-t)^{(1+\alpha)/\beta}},$$
$$(1-\rho)\left[\lambda(y) + \frac{1+\beta}{\beta}\frac{\hat{C}\langle y\rangle^m}{(T-t)^{1/\beta+1}}\right] \le \frac{\bar{C}}{C}\psi(t,y) + \frac{1+\beta}{\beta}\frac{\bar{C}}{C}\frac{\psi(t,y)}{(T-t)^{1/\beta+1}}.$$

Choosing $C > \max\{2m\bar{C} + 2m(m-1)\bar{C}^2, 2^\alpha m^{\alpha+1}\bar{C}^{\alpha+1}, 8\frac{1+\beta}{\beta}\bar{C}\}$, we obtain that

$$\begin{aligned}\mathcal{J}[\chi] >& \frac{L\psi}{(T-t)^{1/\beta}} - \frac{\psi}{\beta(T-t)^{1/\beta+1}} - \frac{C\psi}{(T-t)^{1/\beta}} - C^{\alpha+1}e^{\alpha L(T-t)}\frac{\psi}{(T-t)^{(1+\alpha)/\beta}}\\ &+ \frac{1+\frac{1}{4}\beta}{\beta(T-t)^{1/\beta+1}}\psi - \psi - \frac{\psi}{8(T-t)^{1/\beta+1}}\\ >&\psi\left[\frac{L-C-T^{1/\beta}}{(T-t)^{1/\beta}} + \frac{1-8C^{\alpha+1}e^{\alpha L(T-t)}(T-t)^{1-\alpha/\beta}}{8(T-t)^{1/\beta+1}}\right]\end{aligned}$$

Taking $L > C + T^{1/\beta}$, we get $\mathcal{J}[\chi] > 0$ for all $y \in \mathbb{R}^d$ and $t \in [T-\tau, T)$, where $\tau = \min\{\frac{1}{\alpha L}, (8C^{\alpha+1}e^1)^{(\alpha-\beta)/\alpha}\}$. □

The following lemma is key to the proof of the comparison principle.

Lemma 4.2.4. *Let τ be as in Lemma 4.2.3. The function*

$$\Phi(t,y) := w(t,y) - \chi(t,y)$$

is either nonpositive or attains its supremum at some point $(\bar{t},\bar{y})$ in $[T-\tau,T)\times\mathbb{R}^d$.

Proof. Suppose that the supremum of Φ on $[T-\tau,T)\times\mathbb{R}^d$ is positive and denote by (t_k,y_k) a sequence in $[T-\tau,T)\times\mathbb{R}^d$ approaching the supremum point. For the choice of C in Lemma 4.2.3, $\eta(y) < C\langle y\rangle^m$ for all $y \in \mathbb{R}^d$. Thus, the representation

$$\Phi(t,y) = \frac{\left[\frac{\underline{u}(t,y)(T-t)^{1/\beta}}{\langle y\rangle^{\tilde{m}}} - \frac{\rho\overline{u}(t,y)(T-t)^{1/\beta}}{\langle y\rangle^{\tilde{m}}}\right]\langle y\rangle^{\tilde{m}} - (1-\rho)e^{L(T-t)}C\langle y\rangle^m}{(T-t)^{1/\beta}},$$

along with Condition (4.13) and the fact that $\tilde{m} < m$ yields

$$\limsup_{t\to T}\Phi(t,y) = -\infty, \text{ uniformly on } \mathbb{R}^d.$$

Hence $\lim_k t_k < T$. Furthermore, $\lim_k |y_k| < \infty$ because $w \in \mathcal{SSG}_m^-$. As a result, the supremum is attained at some point $(\bar{t},\bar{y})$ because Φ is upper semicontinuous. This proves the assertion. □

We are now ready to prove the comparison principle.

Proof of Proposition 4.2.1. STEP 1: COMPARISON ON $[T-\tau, T)$. Let τ be as in Lemma 4.2.3. We claim that the function Φ introduced in Lemma 4.2.4 is nonpositive. It then follows that $\underline{u} \leq \overline{u}$ in $[T-\tau, T) \times \mathbb{R}^d$ by letting $\rho \to 1$. In view of Lemma 4.2.4, we just need to consider the case where Φ attains its supremum at some point $(\bar{t}, \bar{y}) \in [T-\tau, T) \times \mathbb{R}^d$. Since χ is smooth and w is a viscosity subsolution to (4.15), we have

$$\begin{aligned} &-\partial_t \chi(\bar{t}, \bar{y}) - \mathcal{L}\chi(\bar{t}, \bar{y}) - (\frac{1-\rho}{2})^{-\alpha} \bar{C}^{\alpha+1} |D\chi|^{\alpha+1} - l(\bar{t}, \bar{y}) w(\bar{t}, \bar{y}) \\ &-(1-\rho)\left[\lambda(\bar{y}) + \frac{1+\beta}{\beta} \frac{\hat{C}\langle y \rangle^m}{(T-t)^{1/\beta+1}}\right] \leq 0. \end{aligned} \tag{4.17}$$

By the mean value theorem and in view of condition (4.14),

$$\begin{aligned} l(t, y) &= \frac{F(y, \underline{u}(t, y)) - F(y, \rho \overline{u}(t, y))}{\underline{u}(t, y) - \rho \overline{u}(t, y)} \mathbb{I}_{\underline{u}(t,y) \neq \overline{u}(t,y)} \\ &\leq \partial_u F(y, \rho \sqrt[\beta]{\frac{\frac{1}{2}\beta + 1}{\beta + 1}} \frac{\eta(y)}{(T-t)^{1/\beta}}) \\ &\leq -\frac{1 + \frac{1}{4}\beta}{\beta(T-t)}. \end{aligned} \tag{4.18}$$

Thus, comparing (4.16) with (4.17) yields

$$l(\bar{t}, \bar{y}) w(\bar{t}, \bar{y}) > -\frac{1 + \frac{1}{4}\beta}{\beta(T-t)} \chi(\bar{t}, \bar{y}) \geq l(\bar{t}, \bar{y}) \chi(\bar{t}, \bar{y}). \tag{4.19}$$

Since $l \leq 0$, we can conclude that $\Phi(\bar{t}, \bar{y}) \leq 0$, and so $\Phi \leq 0$.

STEP 2: COMPARISON ON $[T-\delta, T)$. If $\tau > \delta$, then the proof is finished. Else, we can proceed as follows. From the condition (4.14),

$$\underline{u}(t, y), \overline{u}(t, y) \leq \frac{\hat{C}}{\tau^{1/\beta}} \eta(y), \quad t \in [T-\delta, T-\tau].$$

Since we have shown that $\underline{u}(T-\tau, \cdot) \leq \overline{u}(T-\tau, \cdot)$, an application of our general comparison principle [Proposition A.3.2] shows that $\underline{u} \leq \overline{u}$ on $[T-\delta, T) \times \mathbb{R}^d$. □

We are now going to construct smooth sub- and supersolutions to (4.8) that satisfy the conditions (4.13) and (4.14) of the above proposition. The supersolution will be defined in terms of the function

$$\hat{h}(t, y) := e^{L(T-t)} \langle y \rangle^{\tilde{m}}$$

where $\tilde{m}$ is introduced in condition (F.1), and where the constant L will be determined later. Using the condition (F.1), we can find a constant $C_0 > 0$ such that

$$
\begin{aligned}
&-\partial_t \hat{h}(t,y) - \mathcal{L}\hat{h}(t,y) - 2^\alpha \bar{C}^{\alpha+1}|D\hat{h}(t,y)|^{\alpha+1} - \lambda(y) + \frac{\hat{h}(t,y)^{\beta+1}}{\beta\eta(y)^\beta} \\
&\geq L\hat{h}(t,y) - C_0\hat{h}(t,y) - C_0 e^{\alpha L(T-t)}\hat{h}(t,y) - C_0\hat{h}(t,y) + C_0 e^{\beta L(T-t)}\hat{h}(t,y) \\
&\geq (L-2C_0)\hat{h}(t,y) + C_0 e^{\alpha L(T-t)}\hat{h}(t,y)(e^{(\beta-\alpha)L(T-t)} - 1).
\end{aligned}
$$

Choosing L large enough, we have for $(t,y) \in [0,T] \times \mathbb{R}^d$ that

$$
-\partial_t \hat{h}(t,y) - \mathcal{L}\hat{h}(t,y) - 2^\alpha \bar{C}^{\alpha+1}|D\hat{h}(t,y)|^{\alpha+1} - \lambda(y) + \frac{\hat{h}(t,y)^{\beta+1}}{\beta\eta(y)^\beta} \geq 0. \tag{4.20}
$$

Lemma 4.2.5. *Suppose that Assumptions (L.1)-(L.3), (F.1) and (F.2) hold. Let $\epsilon := 1 - \alpha/\beta$. There exist constants $K > 0, \delta \in (0,T]$ such that*

$$
\check{v}(t,y) := \frac{\eta(y) - \eta(y)\|\frac{\mathcal{L}\eta}{\eta}\|(T-t)}{(T-t)^{1/\beta}}
$$

and

$$
\hat{v}(t,y) := \frac{\eta(y) + \eta(y)K(T-t)^\epsilon}{(T-t)^{1/\beta}} + \hat{h}(t,y)
$$

are a nonnegative classical sub- and supersolution to (4.8) *on $[T-\delta, T) \times \mathbb{R}^d$, respectively. Furthermore, $\check{v}, \hat{v}$ satisfy the conditions* (4.13) *and* (4.14).

Proof. In view of the condition (F.2), the quantity $\|\frac{\mathcal{L}\eta}{\eta}\|$ is well-defined and finite; hence $\delta_0 := 1/\|\frac{\mathcal{L}\eta}{\eta}\| \wedge T > 0$. It has been shown in Chapter 3 that $\check{v}$ is a subsolution to (4.8) on $[T-\delta_0, T) \times \mathbb{R}^d$ when $H = 0$. Since H is nonnegative, we know that $\check{v}$ is still a subsolution on $[T-\delta_0, T) \times \mathbb{R}^d$. We now verify that $\hat{v}$ is a nonnegative classical supersolution to (4.8) on $[T-\delta_1, T) \times \mathbb{R}^d$ for small δ_1. To this end, we first obtain by a direct computation that

$$
\begin{aligned}
&-\partial_t \hat{v}(t,y) - \mathcal{L}\hat{v}(t,y) \\
=& -\frac{\eta(y) + K(1-\beta\epsilon)\eta(y)(T-t)^\epsilon + \beta\mathcal{L}\eta(y)(T-t)\big(1 + K(T-t)^\epsilon\big)}{\beta(T-t)^{(\beta+1)/\beta}} \\
&-\partial_t \hat{h}(t,y) - \mathcal{L}\hat{h}(t,y).
\end{aligned}
$$

Assuming that $K\delta_1^\epsilon \leq 1$ and $\delta_1 \leq 1$, we see that $K(T-t)^\epsilon \leq 1$ and $(T-t)^{1-\epsilon} \leq 1$ for $t \in [T-\delta_1, T)$. Thus,

$$
\begin{aligned}
&-\partial_t \hat{v}(t,y) - \mathcal{L}\hat{v}(t,y) \\
\geq& -\frac{\eta(y) + K(1-\beta\epsilon)\eta(y)(T-t)^\epsilon + 2\beta\bar{C}\eta(y)(T-t)^\epsilon}{\beta(T-t)^{(\beta+1)/\beta}} \\
&-\partial_t \hat{h}(t,y) - \mathcal{L}\hat{h}(t,y).
\end{aligned} \tag{4.21}
$$

Recalling the definition of H and F in (4.7),

$$\begin{aligned}
&- H(y, D\hat{v}(t,y)) \\
&\geq - 2^{\alpha}\bar{C}^{\alpha+1}\frac{|D\eta|^{\alpha+1}[1+K(T-t)^{\epsilon}]^{\alpha+1}}{(T-t)^{(1+\alpha)/\beta}} - 2^{\alpha}\bar{C}^{\alpha+1}|D\hat{h}(t,y)|^{\alpha+1} \\
&\geq - 2^{\alpha}\bar{C}^{\alpha+1}\left\|\frac{|D\eta|^{\alpha+1}}{\eta}\right\|\eta(y)\frac{[1+K(T-t)^{\epsilon}]^{\alpha+1}}{(T-t)^{(1+\alpha)/\beta}} - 2^{\alpha}\bar{C}^{\alpha+1}|D\hat{h}(t,y)|^{\alpha+1} \\
&\geq - 2^{2\alpha+1}\bar{C}^{\alpha+2}\frac{\eta(y)}{(T-t)^{(1+\alpha)/\beta}} - 2^{\alpha}\bar{C}^{\alpha+1}|D\hat{h}(t,y)|^{\alpha+1}.
\end{aligned} \tag{4.22}$$

Applying Bernoulli's inequality in the form $(u+v+w)^{\beta+1} \geq u^{\beta+1}+(\beta+1)u^{\beta}v+w^{\beta+1}$ for $u,v,w \geq 0$ to the term $|\hat{v}(t,y)|^{\beta+1}$ in F, we obtain

$$\begin{aligned}
&- F(y, \hat{v}(t,y)) \\
&\geq - \lambda(y) + \frac{\eta(y)^{\beta+1} + (\beta+1)\eta(y)^{\beta}\eta(y)K(T-t)^{\epsilon}}{\beta\eta(y)^{\beta}(T-t)^{(\beta+1)/\beta}} + \frac{\hat{h}(t,y)^{\beta+1}}{\beta\eta(y)^{\beta}}.
\end{aligned} \tag{4.23}$$

Hence, adding (4.21), (4.22) and (4.23) and using (4.20) yields,

$$\begin{aligned}
&- \partial_t\hat{v}(t,y) - \mathcal{L}\hat{v}(t,y) - H(y, D\hat{v}(t,y)) - F(y,\hat{v}(t,y)) \\
&\geq \eta(y)\frac{(1+\epsilon)K - 2\bar{C} - 2^{2\alpha+1}\bar{C}^{\alpha+2}}{(T-t)^{(1+\alpha)/\beta}} \\
&\quad - \partial_t\hat{h}(t,y) - \mathcal{L}\hat{h}(t,y) - 2^{\alpha}\bar{C}^{\alpha+1}|D\hat{h}(t,y)|^{\alpha+1} - \lambda(y) + \frac{\hat{h}(t,y)^{\beta+1}}{\beta\eta(y)^{\beta}} \\
&\geq \eta(y)\frac{(1+\epsilon)K - 2\bar{C} - 2^{2\alpha+1}\bar{C}^{\alpha+2}}{(T-t)^{(1+\alpha)/\beta}}.
\end{aligned} \tag{4.24}$$

Choosing $K \geq \frac{2\bar{C}+2^{2\alpha+1}\bar{C}^{\alpha+2}}{1+\epsilon}$ and then $\delta_1 = \min\{1, T, \sqrt[\epsilon]{\frac{1}{K}}\}$, we conclude that

$$-\partial_t\hat{v}(t,y) - \mathcal{L}\hat{v}(t,y) - H(y, D\hat{v}(t,y)) - F(y,\hat{v}(t,y)) \geq 0, \quad (t,y) \in [T-\delta_1, T) \times \mathbb{R}^d.$$

Next, we prove that $\check{v}, \hat{v}$ satisfy the asymptotic behaviour (4.13) and (4.14). Recalling the definition of $\check{v}, \hat{v}$ and using the condition (F.1), we have

$$\begin{aligned}
&(T-t)^{1/\beta}\check{v}(t,y) = \eta(y) + \langle y\rangle^{\tilde{m}}O(T-t), && \text{uniformly in } y \text{ as } t \to T. \\
&(T-t)^{1/\beta}\hat{v}(t,y) = \eta(y) + \langle y\rangle^{\tilde{m}}O((T-t)^{\epsilon}), && \text{uniformly in } y \text{ as } t \to T.
\end{aligned}$$

From this, we see that

$$\lim_{t\to T}\frac{\check{v}(t,y)(T-t)^{1/\beta} - \eta(y)}{\langle y\rangle^{\tilde{m}}} = \lim_{t\to T}\frac{\hat{v}(t,y)(T-t)^{1/\beta} - \eta(y)}{\langle y\rangle^{\tilde{m}}} = 0, \tag{4.25}$$

uniformly on $\mathbb{R}^d$, which verifies the condition (4.13). The upper bound in (4.14) can be obtained using the condition (F.1) again. Moreover, for the lower bound in

(4.14), choosing $\delta := \min\{\delta_0(1-\sqrt[\beta]{\frac{\frac{1}{2}\beta+1}{\beta+1}}), \delta_1\}$, we have for $(t,y) \in [T-\delta, T) \times \mathbb{R}^d$ that

$$\hat{v}(t,y)(T-t)^{1/\beta} \geq \check{v}(t,y)(T-t)^{1/\beta} = \eta(y) - \eta(y)\|\frac{\mathcal{L}\eta}{\eta}\|(T-t) \geq \sqrt[\beta]{\frac{\frac{1}{2}\beta+1}{\beta+1}}\eta(y).$$

□

Remark 4.2.6. Due to the presence of the gradient term H, an additional term (4.22) needs to be dominated and thus we make the choice that $\epsilon = 1-\alpha/\beta$. If $H = 0$, we can choose $\epsilon = 1$ as in Chapter 3.

We are now ready to prove the existence result.

Proof of Theorem 4.1.4. In order to apply Perron's method, we set

$$\mathcal{S} = \{u | u \text{ is a subsolution of (4.8) on } [T-\delta, T) \times \mathbb{R}^d \text{ and } u \leq \hat{v}\}.$$

Since $\check{v} \in \mathcal{S}$, the set $\mathcal{S}$ is non-empty. Thus, the function

$$v(t,y) = \sup\{u(t,y) : u \in \mathcal{S}\}$$

is well-defined, belongs to $USC_{\tilde{m}}([T-\delta, T^-] \times \mathbb{R}^d)$ and satisfies that $\check{v} \leq v$. Classical arguments in [CIL92] show that the upper semi-continuous envelope v^* of v is a viscosity subsolution to (4.8) and that the lower semi-continuous envelope v_* of v is a viscosity supersolution to (4.8). Since $\check{v} \leq v_* \leq v^* \leq \hat{v}$, we have for all $(t,y) \in [T-\delta, T) \times \mathbb{R}^d$ that

$$\sqrt[\beta]{\frac{\frac{1}{2}\beta+1}{\beta+1}}\eta(y) \leq v_*(t,y)(T-t)^{1/\beta}, v^*(t,y)(T-t)^{1/\beta} \leq C\langle y\rangle^{\tilde{m}},$$

and

$$\begin{aligned}\frac{\check{v}(t,y)(T-t)^{1/\beta}-\eta(y)}{\langle y\rangle^{\tilde{m}}} \leq \frac{v_*(t,y)(T-t)^{1/\beta}-\eta(y)}{\langle y\rangle^{\tilde{m}}} &\leq \frac{v^*(t,y)(T-t)^{1/\beta}-\eta(y)}{\langle y\rangle^{\tilde{m}}} \\ &\leq \frac{\hat{v}(t,y)(T-t)^{1/\beta}-\eta(y)}{\langle y\rangle^{\tilde{m}}}.\end{aligned}$$

Hence, it follows from (4.25) that

$$\lim_{t\to T}\frac{v_*(t,y)(T-t)^{1/\beta}-\eta(y)}{\langle y\rangle^{\tilde{m}}} = \lim_{t\to T}\frac{v^*(t,y)(T-t)^{1/\beta}-\eta(y)}{\langle y\rangle^{\tilde{m}}} = 0,$$

uniformly on $\mathbb{R}^d$. From our comparison principle [Proposition 4.2.1] we can thus conclude that $v^* \leq v_*$ on $[T-\delta, T) \times \mathbb{R}^d$, which shows that v is the desired viscosity solution to (4.8) that belongs to $C_{\tilde{m}}([T-\delta, T^-] \times \mathbb{R}^d)$.

Next, we find a sub- and supersolution to (4.8) on $[0, T-\delta] \times \mathbb{R}^d$ with terminal value $v(T-\delta, \cdot)$ at $t = T-\delta$. Obviously, 0 is a subsoultion of (4.8). We now conjecture that there exists $\overline{K} > 0$ such that $\overline{w} := \overline{K}\eta + \hat{h}(t,y)$ is a viscosity supersolution to (4.8). In fact, since $v \le \hat{v}$ at $t = T - \delta$, we see that

$$v(T-\delta, y) \le \frac{\bar{C}}{\delta^{1/\beta}}\eta(y) + \hat{h}(T-\delta, y), \quad y \in \mathbb{R}^d.$$

In view of the condition (F.2) and the inequality (4.20), we have that

$$\begin{aligned}
&-\partial_t w(t,y) - \mathcal{L}w(t,y) - H(y, Dw) - F(y, w(t,y)) \\
\ge& -\overline{K}\mathcal{L}\eta(y) - 2^\alpha \bar{C}^{\alpha+1}\overline{K}^{\alpha+1}|D\eta|^{\alpha+1} + \frac{1}{\beta}\overline{K}^{\beta+1}\eta(y) \\
&-\partial_t \hat{h}(t,y) - \mathcal{L}\hat{h}(t,y) - 2^\alpha \bar{C}^{\alpha+1}|Dh(t,y)|^{\alpha+1} - \lambda(y) + \frac{\hat{h}(t,y)^{\beta+1}}{\beta\eta(y)^\beta} \\
\ge& \eta(y)[\frac{1}{\beta}\overline{K}^{\beta+1} - \overline{K}\bar{C} - 2^\alpha \bar{C}^{\alpha+2}\overline{K}^{\alpha+1}] \\
>&0,
\end{aligned}$$

for $\overline{K}$ large enough. Furthermore, $\overline{w}^{\beta+1}/\eta^\beta$ is of polynomial growth of order m. Combining the general comparison principle [Proposition A.3.2] with Perron's method, we obtain a viscosity solution $v \in C_{\tilde{m}}([0, T-\delta] \times \mathbb{R}^d)$. Hence from the comparison principle for continuous viscosity solutions Lemma A.3.4, we get a unique global viscosity solution $v \in C_{\tilde{m}}([0, T^-] \times \mathbb{R}^d)$. □

4.3. Regularity of the viscosity solution

In Section 4.2, we established the existence of a continuous viscosity solution v to (4.8). Unlike in Chapter 3, continuity is not enough to carry out our verification argument [Theorem 4.1.6], due to the dependence of the candidate value function on the gradient. In view of (4.9), the candidate value function, i.e. the viscosity solution should be at least of class $C^{0,1}$. To this end, we proceed as follows. First, we establish the existence of a solution of class $C^{0,1}$ to a modified PDE where the singularity is moved into the nonlinearity. This will provide us with both the necessary regularity properties of the viscosity solution and *a priori* estimates of the solution and its gradient *near the terminal time*. Subsequently, we use a standard link between FBSDEs and viscosity solutions, from which we can derive the differentiability of the viscosity solution on the *whole time interval*.

4.3.1. Mild solution

In what follows, we assume that Assumptions (L.1)-(L.4) and (F.1)-(F.3) hold and that $\beta > 2\alpha$. Recalling the definition that $\epsilon = 1 - \frac{\alpha}{\beta}$ in Lemma 4.2.5, we know that $\epsilon \in (\frac{1}{2}, 1)$. As discussed before, the viscosity solution v constructed in the previous section is of the form

$$v(T-t, y) = \frac{\eta(y) + \tilde{u}(t,y)}{t^{1/\beta}}, \tag{4.26}$$

for some function $\tilde{u}$ that satisfies

$$\tilde{u}(t,y) = O(t^{\epsilon}) \text{ uniformly in } y \text{ as } t \to 0.$$

We choose the following equivalent ansatz:

$$v(T-t,y) = \frac{\eta(y)}{t^{1/\beta}} + \frac{u(t,y)}{t^{1+1/\beta}}, \qquad u(t,y) = O(t^{1+\epsilon}) \text{ uniformly in } y \text{ as } t \to 0. \tag{4.27}$$

It is worth pointing out that if $H = 0$, we can choose $\epsilon = 1$ in (4.26) and (4.27). Plugging the asymptotic ansatz into (4.8) results in a semilinear parabolic equation for u with finite initial condition. The proof of the following lemma is similar to [GHS18, Lemma 4.1] and hence omitted.

Lemma 4.3.1. *If, for some $\delta > 0$, a function $u \in C^{0,1}([0,\delta] \times \mathbb{R}^d)$ satisfies*

$$|u(t,y)| \leq t\eta(y), \quad t \in [0,\delta],\, y \in \mathbb{R}^d, \tag{4.28}$$

and solves the equation

$$\begin{cases} \partial_t u(t,y) = \mathcal{L}u(t,y) + F_0(t,y,u(t,y),Du(t,y)), & t \in (0,\delta]\,, y \in \mathbb{R}^d, \\ u(0,y) = 0, & y \in \mathbb{R}^d, \end{cases} \tag{4.29}$$

where

$$\begin{aligned} F_0(t,y,u,Du) =& t\mathcal{L}\eta(y) + t^p\lambda(y) - \frac{\eta(y)}{\beta}\sum_{k=2}^{\infty}\binom{\beta+1}{k}\left(\frac{u}{t\eta(y)}\right)^k \\ &+ \theta^{\alpha}t^{\epsilon}\left|\sigma^*(y)\left(\frac{Du}{t} + D\eta\right)\right|^{\alpha+1}, \end{aligned}$$

then a local solution $v \in C^{0,1}([T-\delta, T^-] \times \mathbb{R}^d)$ to problem (4.8) is given by

$$v(t,y) = \frac{\eta(y)}{(T-t)^{1/\beta}} + \frac{u(T-t,y)}{(T-t)^{1+1/\beta}}.$$

The case where $H = 0$ has been solved under additional regularity assumptions in [GHS18] using an analytic semigroup approach. Due to the presence of H in our case, we need to choose $\epsilon < 1$, which renders the analysis more complex. In particular, the locally Lipschitz continuity in [GHS18, Lemma 4.5] no longer holds in our case. Instead, we solve equation (4.29) using the weak continuous semigroup approach introduced in [FGS17, Section 4] in order to obtain a $C^{0,1}$ solution.

In a first step we introduce the transition semigroup. Under Assumptions (L.1) and (L.2), the operator

$$P_{t,s}[\varphi](y) = \mathbb{E}[\varphi(Y_s^{t,y})], \quad \varphi \in C_b(\mathbb{R}^d),\, 0 \leq t \leq s$$

is well-defined and satisfies the Markov property $P_{t,r} = P_{t,s}P_{s,r}$ for $0 \leq t \leq s \leq r$. Since b and σ are independent of the time variable,

$$P_{t,s}[\varphi](y) = P_{0,s-t}[\varphi](y).$$

For convenience, we denote

$$P_t[\varphi](y) = \mathbb{E}[\varphi(Y_t^{0,y})], \quad \varphi \in C_b(\mathbb{R}^d). \tag{4.30}$$

For every $\varphi \in C_b(\mathbb{R}^d)$,

$$|P_t[\varphi](y)| \le \|\varphi\|, \quad (t,y) \in [0,T] \times \mathbb{R}^d.$$

Furthermore, from [FGS17, Theorem 4.65], we have the following proposition.

Proposition 4.3.2. *Suppose that Assumptions (L.1)-(L.4) hold and let $\varphi \in C_b(\mathbb{R}^d)$. Then for every $0 \le t \le T$, the function $y \to P_t[\varphi](y)$ is continuously differentiable on $\mathbb{R}^d$. Moreover, there exists a constant $M > 0$ such that for every $\varphi \in C_b(\mathbb{R}^d)$ and for $0 \le t \le T$,*

$$|DP_t[\varphi](y)| \le \frac{M}{t^{1/2}}\|\varphi\|, \quad y \in \mathbb{R}^d. \tag{4.31}$$

Next, we introduce the notion of a mild solution of our modified PDE.

Definition 4.3.3. We say that a function $u : [0,\delta] \times \mathbb{R}^d \to \mathbb{R}$ is a mild solution of the PDE (4.29) if the following conditions are satisfied:

(i) $u \in C_b^{0,1}([0,\delta] \times \mathbb{R}^d)$.

(ii) for every $t \in [0,T]$ and $y \in \mathbb{R}^d$,

$$u(t,y) = \int_0^t P_{t-s}[F_0(s,\cdot,u(s,\cdot),Du(s,\cdot))](y)ds.$$

We prove the existence of a mild solution to (4.29) by a contraction argument. To this end, we need to choose an appropriate weighted norm on $C_b^{0,1}([0,\delta] \times \mathbb{R}^d)$ to cope with the singularity in F_0. Recalling the ansatz (4.27) and the property (4.31), we consider the space

$$\Sigma := \Big\{u \in C_b^{0,1}([0,\delta] \times \mathbb{R}^d) : \|u(t,\cdot)\| + \|t^{1/2}Du(t,\cdot)\| = O(t^{1+\epsilon}) \ \textit{as}\ t \to 0\Big\},$$

endowed with the weighted norm

$$\|u\|_\Sigma = \sup_{(t,y)\in(0,\delta]\times\mathbb{R}^d} \left(\frac{|u(t,y)|}{t^{1+\epsilon}} + \frac{|Du(t,y)|}{t^{1/2+\epsilon}}\right).$$

It is easy to verify that the vector space Σ endowed with the norm $\|\cdot\|_\Sigma$ is a Banach space.

Lemma 4.3.4. *Suppose that $\beta > 2\alpha$ and that Assumptions (L.1)-(L.4) and (F.1)-(F.3) hold. Let $R > 0$ and $\delta \in (0, \sqrt[\epsilon-\frac{1}{2}]{\underline{c}/R} \wedge 1]$. Define the closed ball*

$$\overline{B}_\Sigma(R) := \{u \in \Sigma : ||u||_\Sigma \le R\}.$$

For every $u \in \overline{B}_\Sigma(R)$, the function

$$f_0(t,y) := F_0(t,y,u(t,y),Du(t,y))$$

is continuous.

Proof. For $u \in \overline{B}_\Sigma(R)$, we may decompose $f_0(t,y)$ in the following way:

$$f_0(t,y) = t\mathcal{L}\eta(y) + t^p\lambda(y) - (p-1)\eta(y)g_0(t,y) + \theta^\alpha t^\epsilon g_1(t,y). \tag{4.32}$$

where

$$g_0(t,y) = \sum_{k=2}^{\infty} \binom{\beta+1}{k} \left(\frac{u(t,y)}{t\eta(y)}\right)^k$$

and

$$g_1(t,y) = \left|\sigma^*(y)\left(\frac{Du(t,y)}{t} + D\eta(y)\right)\right|^{\alpha+1}.$$

The assumption $\delta \leq \sqrt[\epsilon-\frac{1}{2}]{\underline{c}/R}$ guarantees that the series converges since then

$$\left|\frac{u(t,y)}{t\eta(y)}\right| \leq \frac{t^{1+\epsilon}R}{t\underline{c}} \leq \frac{\delta^\epsilon R}{\underline{c}} \leq 1, \quad t \in [0,\delta], y \in \mathbb{R}^d.$$

Moreover,

$$\left|\frac{Du(t,y)}{t}\right| \leq \frac{t^{\frac{1}{2}+\epsilon}R}{t} \leq \delta^{\epsilon-\frac{1}{2}}R \leq \underline{c}, \quad t \in [0,\delta], y \in \mathbb{R}^d. \tag{4.33}$$

In view of (4.32) it is sufficient to prove that g_0 and g_1 are continuous in t, uniformly with respect to y on every compact subset of $\mathbb{R}^d$. In fact, by the mean value theorem and the triangle inequality, we have for $0 \leq t \leq s \leq \delta, y \in \mathbb{R}^d$ that

$$\begin{aligned}
&|g_1(t,y) - g_1(s,y)| \\
\leq& \left| \left|\sigma^*(y)\left(\frac{Du(t,y)}{t} + D\eta(y)\right)\right|^{\alpha+1} - \left|\sigma^*(y)\left(\frac{Du(s,y)}{s} + D\eta(y)\right)\right|^{\alpha+1} \right| \\
\leq& (\alpha+1)\bar{C}^{\alpha+1}(\underline{c}+\bar{C})^\alpha \left|\frac{Du(t,y)}{t} - \frac{Du(s,y)}{s}\right|.
\end{aligned}$$

In order to establish the continuity of g_0, notice that for $0 \leq t \leq s \leq \delta, y \in \mathbb{R}^d$ and $k \geq 2$ it holds that

$$\begin{aligned}
&\left|\left(\frac{u(t,y)}{t\eta(y)}\right)^k - \left(\frac{u(s,y)}{s\eta(y)}\right)^k\right| \\
\leq& \frac{1}{\underline{c}^k}\left|\frac{u(t,y)}{t} - \frac{u(s,y)}{s}\right| \sum_{l=0}^{k-1} \left|\frac{u(t,y)}{t}\right|^l \left|\frac{u(s,y)}{s}\right|^{k-1-l} \\
\leq& \frac{R^{k-1}}{\underline{c}^k}\left|\frac{u(t,y)}{t} - \frac{u(s,y)}{s}\right| \sum_{l=0}^{k-1} t^{\epsilon l} s^{\epsilon(k-1-l)} \\
\leq& \frac{kR^{k-1}}{\underline{c}^k}\left|\frac{u(t,y)}{t} - \frac{u(s,y)}{s}\right| s^{(k-1)\epsilon} \\
\leq& \frac{k}{\underline{c}}\left(\frac{Rs^\epsilon}{\underline{c}}\right)^{k-1}\left|\frac{u(t,y)}{t} - \frac{u(s,y)}{s}\right|.
\end{aligned} \tag{4.34}$$

Using the identity $k\binom{\beta+1}{k} = (\beta+1)\binom{\beta}{k-1}$, we get that

$$|g_0(t,y) - g_0(s,y)| \leq (\beta+1)\max\{2^\beta - 1, \beta\}\frac{Rs^\epsilon}{\underline{c}^2}\left|\frac{u(t,y)}{t} - \frac{u(s,y)}{s}\right|.$$

Hence the claim follows from the fact that the maps $(t,y) \mapsto \frac{u(t,y)}{t}, \frac{Du(t,y)}{t}$ are continuous on $[0,\delta]\times\mathbb{R}^d$. ☐

The following lemma can be established using similar arguments as above.

Lemma 4.3.5. *Suppose that $\beta > 2\alpha$ and that Assumptions (L.1)-(L.4) and (F.1)-(F.3) hold. For every $R > 0$ there exists a positive constant L independent of $\delta \in (0, \sqrt[\epsilon-\frac{1}{2}]{\underline{c}/R}]$ such that for $u, v \in \overline{B}_\Sigma(R)$, $t \in [0,\delta], y \in \mathbb{R}^d$,*

$$\begin{aligned}&|F_0(t,y,u(t,y),Du(t,y)) - F_0(t,y,v(t,y),Dv(t,y))|\\ \leq& Lt^\epsilon\left(\frac{|u(t,y)-v(t,y)|}{t} + \frac{|Du(t,y)-Dv(t,y)|}{t}\right).\end{aligned}$$

We are now ready to carry out the fixed point argument.

Theorem 4.3.6. *Let $\beta > 2\alpha$. Under Assumptions (L.1)-(L.4) and (F.1)-(F.3), there exists a constant $\delta > 0$ such that Equation* (4.29) *admits a mild solution $u \in C_b^{0,1}([0,\delta]\times\mathbb{R}^d)$.*

Proof. Let us define the operator

$$\Gamma[u](t,y) := \int_0^t P_{t-s}[F_0(s,\cdot,u(s,\cdot),Du(s,\cdot))](y)ds \tag{4.35}$$

Step 1: the map Γ is well defined on $\overline{B}_\Sigma(R)$. Let $u \in \overline{B}_\Sigma(R)$. By Lemma 4.3.4 and [FGS17, Proposition 4.67][2], we see that

$$\Gamma[u] \in C_b([0,\delta]\times\mathbb{R}^d), \quad D\Gamma[u] \in C_b((0,\delta]\times\mathbb{R}^d).$$

In order to see the continuity of $D\Gamma[u]$ at $t = 0$, we differentiate (4.35) to obtain that

$$D\Gamma[u](t,y) = \int_0^t DP_{t-s}[F_0(s,\cdot,u(s,\cdot),Du(s,\cdot))](y)ds, \quad (t,y) \in [0,\delta]\times\mathbb{R}^d. \tag{4.36}$$

By Proposition 4.3.2,

$$|D\Gamma[u](t,y)| \leq \int_0^t M\frac{\|f_0\|}{(t-s)^{1/2}}ds = \sqrt{t}M\|f_0\|.$$

From this, we conclude that the map $(t,y) \mapsto D\Gamma[u](t,y)$ belongs to $C_b([0,\delta]\times\mathbb{R}^d)$.

[2]The strong continuity in this proposition is equivalent to the standard continuity in finite-dimensional space.

Step 2: contraction property of Γ on $\overline{B}_\Sigma(R)$ for a suitable choice of R, δ. Let

$$B(a,b) := \int_0^1 r^{a-1}(1-r)^{b-1} dr$$

be the Beta function with $a, b > 0$. We choose

$$R = 2(1 + MB_0)\left(\|\mathcal{L}\eta\| + \|\lambda\| + \|\sigma^* D\eta\|^{\alpha+1}\right),$$

and

$$\delta = \min\{ \sqrt[\epsilon-\frac{1}{2}]{\underline{c}/R}, \sqrt[\epsilon-\frac{1}{2}]{1/\big(2L(1+MB_1)\big)}, 1\},$$

where $L > 0$ is the Lipschitz constant given by Lemma 4.3.5 and

$$B_0 := B(1+\epsilon, \frac{1}{2}), \quad B_1 := B(2\epsilon + \frac{1}{2}, \frac{1}{2}).$$

Let $u, v \in \overline{B}_\Sigma(R)$. By Lemma 4.3.5, we have for $(t,y) \in [0,\delta] \times \mathbb{R}^d$ that

$$\begin{aligned}
&|\Gamma[u](t,y) - \Gamma[v](t,y)| \\
=&\left|\int_0^t P_{t-s}[F_0(s,\cdot,u(s,\cdot),Du(s,\cdot)) - F_0(s,\cdot,v(s,\cdot),Dv(s,\cdot))](y) ds\right| \\
\le&\int_0^t \|F_0(s,y,u(s,\cdot),Du(s,\cdot)) - F_0(s,\cdot,v(s,\cdot),Dv(s,\cdot))\| \, ds \\
\le&\int_0^t Ls^\epsilon \left(\frac{\|u(s,\cdot)-v(s,\cdot)\|}{s} + \frac{\|Du(s,\cdot)-Dv(s,\cdot)\|}{s})\right) ds \\
=&\int_0^t L \left(s^{2\epsilon}\frac{\|u(s,\cdot)-v(s,\cdot)\|}{s^{1+\epsilon}} + s^{2\epsilon-1/2}\frac{\|Du(s,\cdot)-Dv(s,\cdot)\|}{s^{1/2+\epsilon}}\right) ds \\
\le& Lt^{2\epsilon+1/2}\|u-v\|_\Sigma.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&|D\Gamma[u](t,y) - D\Gamma[v](t,y)| \\
=&\left|\int_0^t DP_{t-s}[F_0(s,\cdot,u(s,\cdot),Du(s,\cdot)) - F_0(s,\cdot,v(s,\cdot),Dv(s,\cdot))](y) ds\right| \\
\le& M\int_0^t \frac{1}{(t-s)^{1/2}} \|F_0(s,y,u(s,\cdot),Du(s,\cdot)) - F_0(s,\cdot,v(s,\cdot),Dv(s,\cdot))\| \, ds \\
\le&\int_0^t ML\frac{1}{(t-s)^{1/2}}\left(s^{2\epsilon-1/2}\|u-v\|_\Sigma\right) ds \\
\le& MLB_1 t^{2\epsilon}\|u-v\|_\Sigma.
\end{aligned}$$

Hence

$$\|\Gamma[u] - \Gamma[v]\|_\Sigma \le \frac{1}{2}\|u-v\|_\Sigma.$$

Step 3: Γ maps $\overline{B}_\Sigma(R)$ into itself. Note that $s^k \le 1$ for all $k > 0$ and $s \in [0,\delta]$ since $\delta \le 1$. Hence, it holds for every $t \in [0,\delta]$ that

$$|\Gamma[0](t,y)| = \left|\int_0^t P_{t-s}[F_0(s,\cdot,0,0)](y) ds\right|$$

$$\leq \int_0^t \|s\mathcal{L}\eta + s^p\lambda + \theta^\alpha s^\epsilon|\sigma^* D\eta|^{\alpha+1}\|\, ds$$
$$\leq t^{1+\epsilon}(\|\mathcal{L}\eta\| + \|\lambda\| + \|\sigma^* D\eta\|^{\alpha+1}\|),$$

and

$$\begin{aligned}|D\Gamma[0](t,y)| &= \left|\int_0^t DP_{t-s}[F_0(s,\cdot,0,0)](y)ds\right| \\ &\leq \int_0^t \frac{1}{(t-s)^{1/2}} M\|s\mathcal{L}\eta + s^p\lambda + \theta^\alpha s^\epsilon|\sigma^* D\eta|^{\alpha+1}\|\, ds \\ &\leq t^{1+\epsilon-1/2} M B_0(\|\mathcal{L}\eta\| + \|\lambda\| + \|\sigma^* D\eta\|^{\alpha+1}).\end{aligned}$$

Thus,

$$\|\Gamma[u]\|_\Sigma \leq \|\Gamma[u] - \Gamma[0]\|_\Sigma + \|\Gamma[0]\|_\Sigma \leq R.$$

Hence, Γ is a contraction from $\overline{B}_\Sigma(R)$ to itself and has a unique fixed point u in $\overline{B}_\Sigma(R)$. □

4.3.2. Gradient estimate of the viscosity solution

It can be easily proved that the mild solution $u \in C_b^{0,1}([0,\delta] \times \mathbb{R}^d)$ obtained in Theorem 4.3.6 is also a viscosity solution of (4.29) on $[0,\delta] \times \mathbb{R}^d$. Thus

$$w(t,y) := \frac{\eta(y)}{(T-t)^{1/\beta}} + \frac{u(T-t,y)}{(T-t)^{1+1/\beta}}$$

is a viscosity solution of (4.8) in $C_b^{0,1}([T-\delta,T^-] \times \mathbb{R}^d)$. By Lemma A.3.4, $v = w$ on $[T-\delta,T) \times \mathbb{R}^d$. In view of (4.33) and the boundedness of $D\eta$ derived from (F.2) and (F.3), we see that there exits a constant $C > 0$ such that for $(t,y) \in [T-\delta,T) \times \mathbb{R}^d$,

$$|Dv(t,y)| \leq \frac{C}{(T-t)^{1/\beta}}. \tag{4.37}$$

It remains to establish an *a priori* estimate for Dv on $[0,T-\delta] \times \mathbb{R}^d$. To this end, we introduce a family of quadratic FBSDE systems whose terminal value at time $T_0 \in (0,T)$ is given by $v(T_0,\cdot)$. The first component of the solution to the BSDE is given in terms of the viscosity solution. The differentiability of the viscosity solution can then be inferred from the differentiability of the corresponding BSDE.

Lemma 4.3.7. *Suppose that $\beta > 2\alpha$ and that Assumptions (L.1)-(L.4) and (F.1)-(F.3) hold. There exists processes $(U^{t,y}, Z^{t,y}) \in S^\infty_{\mathcal{F}}(t,T^-;\mathbb{R}) \times H^q_{\mathcal{F}}(t,T^-;\mathbb{R}^{1\times\tilde{d}})$ for all $q \geq 2$ satisfying $U^{t,y}_t = v(t,y)$ and for any $t \leq r \leq s < T$,*

$$U^{t,y}_r = U^{t,y}_s + \int_r^s F(Y^{t,y}_\rho, U^{t,y}_\rho) + \theta^\alpha |Z^{t,y}_\rho|^{1+\alpha} d\rho - \int_r^s Z^{t,y}_\rho dW_\rho. \tag{4.38}$$

Proof. For $T_0 \in (0,T)$, we consider the PDE

$$\begin{cases} -\partial_t w(t,y) - \mathcal{L}w(t,y) - H(Dw(t,y)) - f(t,y) = 0, & (t,y) \in [0,T_0) \times \mathbb{R}^d, \\ w(T_0,y) = v(T_0,y), & y \in \mathbb{R}^d. \end{cases} \tag{4.39}$$

where $f(t,y) := F(y, v(t,y))$ for $(t,y) \in [0,T_0] \times \mathbb{R}^d$, and the forward-backward system

$$\begin{cases} dY_s^{t,y} = b(Y_s^{t,y})ds + \sigma(Y_s^{t,y})dW_s, & s \in [t,T_0], \\ dU_s^{t,y} = -f(s, Y_s^{t,y}) - \theta^\alpha |Z_s^{t,y}|^{1+\alpha} ds + Z_s^{t,y} dW_s, & s \in [t,T_0], \\ \;Y_t^{t,y} = y, U_{T_0}^{t,y} = v(T_0, Y_{T_0}^{t,y}). \end{cases} \tag{4.40}$$

From [IRZ10, Theorem 1], the system (4.40) admits a unique solution

$$(Y_s^{t,y}, U_s^{t,y}, Z_s^{t,y})_{t \le s \le T_0} \in S^2_{\mathcal{F}}(t,T_0;\mathbb{R}^d) \times S^\infty_{\mathcal{F}}(t,T_0;\mathbb{R}) \times H^q_{\mathcal{F}}(t,T_0;\mathbb{R}^{1\times \tilde{d}})$$

and $\int_t^\cdot Z_s dW_s$ is a BMO martingale. Furthermore, the map $(t,y) \mapsto U_t^{t,y}$ defines a viscosity solution of (4.39) by [BH07, Proposition 8]. Hence it follows from the comparison principle [Proposition A.3.1] that $U_t^{t,y} = v(t,y)$ on $[0,T_0] \times \mathbb{R}^d$. As a result, we have for any $r \in [t,T_0]$ that $0 \le U_r^{t,y} = v(r, Y_r^{t,y})$. Thus $(U_s^{t,y}, Z_s^{t,y})_{t \le s \le T_0}$ is also a solution to the following BSDE:

$$\begin{cases} dU_s^{t,y} = -F(Y_s^{t,y}, U_s^{t,y}) - \theta^\alpha |Z_s^{t,y}|^{1+\alpha} ds + Z_s^{t,y} dW_s, & s \in [t,T_0] \\ \;U_{T_0}^{t,y} = v(T_0, Y_{T_0}^{t,y}). \end{cases}$$

Since T_0 is arbitrary, we obtain a solution to the BSDE (4.38) on $[0,T)$. □

Proposition 4.3.8. *Let $\beta > 2\alpha$. Under Assumptions (L.1)-(L.4) and (F.1)-(F.3), the function $v(t,\cdot)$ is continuously differentiable for any $t \in [0,T)$. In addition, for every $y \in \mathbb{R}^d$, $0 \le t \le r < T$,*

$$Z_r^{t,y} = Dv(r, Y_r^{t,y})\sigma(Y_r^{t,y}),$$

where $Z^{t,y}$ is the second component of the solution to the BSDE (4.38)*, and*

$$|Z_r^{t,y}| \le \begin{cases} \dfrac{C}{(T-r)^{1/\beta}}, & r \in [T-\delta, T); \\ C\left(1 + \dfrac{1}{\delta^{1/\beta}}\right), & r \in [t, T-\delta]. \end{cases} \tag{4.41}$$

Proof. Since we have proved that $v(r,\cdot)$ is differentiable for $r \in [T-\delta,T)$, it follows by Itô's formula that $Z_r^{t,y} = Dv(r, Y_r^{t,y})\sigma(Y_r^{t,y})$, for $r \in [T-\delta,T)$. The estimate on $[T-\delta,T)$ can thus be obtained from (L.3) and (4.37).

Next, we extend the domain of the solution by setting $Y_s^{t,y} = y$ for $s \in [0,t)$ and then consider the BSDE (4.38) on $[0,T-\delta]$. From [BC08, Proposition 12], the map $(t,y) \mapsto (U_\cdot^{t,y}, Z_\cdot^{t,y})$ belongs to $C^{0,1}([0,T-\delta]\times\mathbb{R}^d; S^\infty_{\mathcal{F}} \times H^q_{\mathcal{F}})$. Moreover, by [BC08, Theorem 15], for $0 \le t \le r \le T-\delta$, the map $y \mapsto U_t^{t,y} = v(t,y)$ is differentiable and $Z_r^{t,y} = Dv(r, Y_r^{t,y})\sigma(Y_r^{t,y})$. The estimate on $Z^{t,y}$ can be obtained using the similar argument in the proof of [Ric12, Theorem 3.6]. We sketch the proof for the reader's convenience. Denote the generator in (4.38) by g, differentiating (4.38) yields

$$\begin{aligned} DU_r^{t,y} =& Dv(T-\delta, Y_{T-\delta}^{t,y})DY_{T-\delta}^{t,y} - \int_r^{T-\delta} (DZ_\rho^{t,y})^* dW_\rho \\ &+ \int_r^{T-\delta} \partial_y g \cdot DY_\rho^{t,y} + \partial_u g \cdot DU_\rho^{t,y} + \partial_z g \cdot DZ_\rho^{t,y}\, d\rho \end{aligned}$$

where

$$\begin{aligned} \partial_y g &= D\lambda(Y^{t,y}) + D\eta(Y^{t,y}) \left(\frac{U^{t,y}}{\eta(Y^{t,y})}\right)^{\beta+1}; \\ \partial_u g &= -\frac{\beta+1}{\beta}\left(\frac{U^{t,y}}{\eta(Y^{t,y})}\right)^{\beta}; \\ \partial_z g &= (\alpha+1)\theta^\alpha |Z^{t,y}|^{\alpha-1} Z^{t,y}, \end{aligned}$$

and $Z_r^{t,y} = DU_r^{t,y}(DY_r^{t,y})^{-1}\sigma(Y_r^{t,y})$. Furthermore, since $\int_t^\cdot Z_\rho^{t,y} dW_\rho$ is BMO and

$$|\partial_z g| \le C(1 + |Z^{t,y}|),$$

the process $\int_t^\cdot \partial_z g dW_\rho$ is BMO. We can thus apply Girsanov's theorem to see that

$$\tilde{W}_r = W_r - \int_t^r \partial_z g\, d\rho$$

is a Brownian motion under the probability Q with

$$\frac{dQ}{d\mathbb{P}} = \mathcal{E}\left(\int_t^\cdot \partial_z g\, dW_\rho\right)_{T-\delta}.$$

We obtain that

$$DU_r^{t,y} = \mathbb{E}^Q\left[e^{\int_r^{T-\delta}\partial_u g d\rho} Dv(T-\delta, Y_{T-\delta}^{t,y})DY_{T-\delta}^{t,y} + \int_r^{T-\delta} e^{\int_r^\rho \partial_u g d\rho}\partial_y g \cdot DY_\rho^{t,y}\, d\rho\right]$$

and hence

$$|Z_r^{t,y}| \le C\big(1 + \frac{1}{\delta^{1/\beta}}\big) \cdot \mathbb{E}^Q\left[\sup_{r \le \rho \le T-\delta} |DY_\rho^{t,y}(DY_r^{t,y})^{-1}|\right]. \tag{4.42}$$

by the boundedness of $\partial_u g, \partial_y g$ and the estimate (4.37). Let us denote

$$\mathcal{E}_{r,T-\delta} := \mathcal{E}\left(\int_r^{T-\delta} \partial_z g\, dW_\rho\right).$$

Since $\int_t^\cdot Z_\rho dW_\rho$ is BMO, there exists $q > 1$ such that $\mathbb{E}[\mathcal{E}^q_{r,T-\delta}] < +\infty$. Moreover, $DY_\rho(DY_r)^{-1}$ solves the SDE

$$\tilde{Y}^{t,y}_\rho = I_d + \int_r^\rho Db(Y^{t,y}_\zeta)\tilde{Y}^{t,y}_\zeta d\zeta + \sum_{i=1}^{\tilde{d}} \int_r^\rho D\sigma^i(Y^{t,y}_\zeta)\tilde{Y}^{t,y}_\zeta dW^i_\zeta.$$

By classical SDE estimates, we have that

$$\begin{aligned}
&\mathbb{E}^Q\left[\sup_{r\le\rho\le T-\delta}|DY^{t,y}_\rho(DY^{t,y}_r)^{-1}|\right]\\
\le&\mathbb{E}\left[\mathcal{E}^q_{r,T-\delta}\right]^{1/q}\mathbb{E}\left[\sup_{r\le\rho\le T-\delta}|DY^{t,y}_\rho(DY^{t,y}_r)^{-1}|^{q'}\right]^{1/q'}\\
\le&C,
\end{aligned}$$

where q' is the conjugate of q. Putting this inequality into (4.42) completes the proof. $\square$

4.4. Verification

This section is devoted to the verification argument. We first prove admissibility of the strategy ξ^* by using the estimates of the nonnegative viscosity solution v derived from the proof of Theorem 4.1.4. Since the optimal density ϑ^* takes values in an unbounded set, one needs an additional argument to guarantee that the corresponding stochastic exponential is a true martingale. Subsequently, we show that (ξ^*, ϑ^*) is a saddle point of the cost function and is indeed optimal.

Lemma 4.4.1. *The feedback controls ξ^* given by (4.9) is admissible, and the portfolio process $(X^*_s)_{s\in[t,T]}$ is monotone.*

The proof is similar to Lemma 3.2.7 and hence omitted. The following lemma shows that for any $\xi \in \mathcal{A}(t,x)$ the expected residual costs vanish as $s \to T$ under a particular class of equivalent measure.

Lemma 4.4.2. *For every $\xi \in \mathcal{A}(t,x)$ and every $Q \in \mathcal{Q}$ satisfying*

$$\mathbb{E}\left[e^{q\int_t^T|\vartheta_r|^2\,dr}\right] < \infty, \quad \textit{for every } q > 0,$$

it holds that

$$\mathbb{E}_Q\left[v(s, Y^{t,y}_s)|X^\xi_s|^p\right] \longrightarrow 0, \quad s \to T. \tag{4.43}$$

Proof. Set $\pi_s = \mathcal{E}(\int_t^s \vartheta_r dW_r)$. For $k > 1, s \in [t,T]$,

$$\mathbb{E}\left[(\pi_s)^k\right] \le \left(\mathbb{E}\left[\mathcal{E}(2k\int_t^s\vartheta_r dW_r)\right]\right)^{1/2}\cdot\left(\mathbb{E}\left[e^{(2k^2-k)\int_t^s|\vartheta_r|^2\,dr}\right]\right)^{1/2} < \infty.$$

Using the similar argument as in Chapter 3, we obtain

$$|X_s^\xi|^p \le C(T-s)^{1/\beta} E\left[\int_s^T |\xi_r|^p\,dr\middle|\mathcal{F}_s\right],$$

and

$$v(s, Y_s^{t,y}) \le \frac{C}{(T-s)^{1/\beta}}.$$

Therefore,

$$\begin{aligned}\mathbb{E}_Q\left[v(s,Y_s^{t,y})|X_s^\xi|^p\right] &= \mathbb{E}\left[\pi_s v(s,Y_s^{t,y})|X_s^\xi|^p\right]\\ &\le C\mathbb{E}\left[\pi_s \int_s^T |\xi_r|^p\,dr\right]\\ &\le C\left((T-s)\mathbb{E}\left[(\pi_s)^2\right]\mathbb{E}\left[\int_s^T |\xi_r|^{2p}\,dr\right]\right)^{1/2}.\end{aligned}$$

Letting $s \to T$, the desired result (4.43) follows since $\xi \in L^{2p}_{\mathcal{F}}(0,T;\mathbb{R})$. □

Now we are ready to carry out the verification argument. We will show that $v(\cdot,\cdot)|\cdot|^p$ is indeed equal to the value function of our control problem and that the candidate strategy is optimal on the whole time interval.

Proof of Theorem 4.1.6. For fixed $t \le s < T$, by Lemma 4.3.7 we have that

$$U_t^{t,y} = U_s^{t,y} + \int_t^s \left(F(Y_r^{t,y}, U_r^{t,y}) + \theta^\alpha |Z_r^{t,y}|^{1+\alpha}\right)dr - \int_t^s Z_r^{t,y}\,dW_r.$$

This allows us to apply to $U_r^{t,y}|X_r^\xi|^p$ the integration by parts formula on $[t,s]$ and to get that

$$\begin{aligned}U_t^{t,y}|x|^p =& U_s^{t,y}|X_s^\xi|^p + \int_t^s \left\{(F(Y_r^{t,y}, U_r^{t,y}) + \theta^\alpha |Z_r^{t,y}|^{1+\alpha})|X_r^\xi|^p\right.\\ &\left.+ p\xi_r U_r^{t,y}\operatorname{sgn}(X_r^\xi)|X_r^\xi|^{p-1})\right\}dr - \int_t^s Z_r^{t,y}|X_r^\xi|^p\,dW_r.\end{aligned}$$

Denote $W_r^\vartheta = W_r - \int_t^r \vartheta_\rho d\rho$. Thus,

$$\begin{aligned}U_t^{t,y}|x|^p =& U_s^{t,y}|X_s^\xi|^p + \int_t^s \left\{(F(Y_r^{t,y}, U_r^{t,y}) + \theta^\alpha |Z_r^{t,y}|^{1+\alpha} - \vartheta_r Z_r^{t,y})|X_r^\xi|^p\right.\\ &\left.+ p\xi_r U_r^{t,y}\operatorname{sgn}(X_r^\xi)|X_r^\xi|^{p-1})\right\}dr - \int_t^s Z_r^{t,y}|X_r^\xi|^p\,dW_r^\vartheta. \qquad (4.44)\end{aligned}$$

In what follows, we show that (ξ^*, ϑ^*) is a saddle point of the functional $\tilde{J}$, i.e.

$$\tilde{J}(t,y,x;\xi^*,\vartheta) \le \tilde{J}(t,y,x;\xi^*,\vartheta^*) \le \tilde{J}(t,y,x;\xi,\vartheta^*).$$

Step 1: $\tilde{J}(t,y,x;\xi^*,\vartheta^*) \leq \tilde{J}(t,y,x;\xi,\vartheta^*)$ for every ξ. Set $\pi_s^* = \mathcal{E}(\int_t^s \vartheta_r^* dW_r)$. From the definition of ϑ^* in (4.9), we see that $|\vartheta_r^*| \leq (1+\alpha)\theta^\alpha |Z_r^{t,y}|^\alpha$. Using the estimate in (4.41),

$$\begin{aligned}\int_t^T |\vartheta_s^*|^2 ds &\leq \int_{T-\delta}^T |\vartheta_s^*|^2 ds + \int_t^{T-\delta} |\vartheta_s^*|^2 ds \\ &\leq (1+\alpha)^2\theta^{2\alpha}\left(\int_{T-\delta}^T \frac{C^\alpha}{(T-s)^{2\alpha/\beta}} ds + \int_t^{T-\delta} C^{2\alpha}(1+\frac{1}{\delta^{1/\beta}})^{2\alpha} ds\right) \\ &\leq (1+\alpha)^2\theta^{2\alpha}\left(C^\alpha \delta^{1-2\alpha/\beta} + TC^{2\alpha}(1+\frac{1}{\delta^{1/\beta}})^{2\alpha}\right) < +\infty.\end{aligned} \tag{4.45}$$

Hence $\mathbb{E}[(\pi_s^*)^k] < +\infty$ for every $k > 1$. This allows us to show that the stochastic integral in (4.44) is a Q^*-martingale. Since $Z^{t,y}$ is bounded away from the terminal time and

$$\mathbb{E}[\sup_{t \leq r \leq s} |X_r^\xi|^{2p}] \leq C\left(|x|^{2p} + \mathbb{E}[\int_t^T |\xi_r|^{2p}\, dr\right),$$

we have that

$$\begin{aligned}&\mathbb{E}_{Q^*}\left[\int_t^s |Z_r^{t,y}|^2 |X_r^\xi|^{2p}\, dr\right]^{1/2} \\ \leq& C\mathbb{E}\left[(\pi_s^*)^2 \int_t^s |X_r^\xi|^{2p}\, dr\right]^{1/2} \\ \leq& C\mathbb{E}\left[\frac{(\pi_s^*)^2}{2} + \frac{T\sup_{t\leq r\leq s}|X_r^\xi|^{2p}}{2}\right] \\ <& +\infty.\end{aligned}$$

Set

$$C(y,x,\xi,\vartheta) := c(y,x,\xi) - \frac{1}{\theta} a|\vartheta|^m |x|^p.$$

Thus,

$$\begin{aligned}&U_t^{t,y}|x|^p \\ =&\mathbb{E}_{Q^*}\left[U_s^{t,y}|X_s^\xi|^p\right] + \mathbb{E}_{Q^*}\left[\int_t^s C(Y_r^{t,y}, X_r^\xi, \xi_r, \vartheta_r^*)\, dr\right] \\ &+ \mathbb{E}_{Q^*}\left[\int_t^s \left\{F(Y_r^{t,y}, U_r^{t,y})|X_r^\xi|^p + p\xi_r U_r^{t,y} \operatorname{sgn}(X_r^\xi)|X_r^\xi|^{p-1} - c(Y_r^{t,y}, X_r^\xi, \xi_r)\right\} dr\right] \\ \leq& \mathbb{E}_{Q^*}\left[U_s^{t,y}|X_s^\xi|^p\right] + \mathbb{E}_{Q^*}\left[\int_t^s C(Y_r^{t,y}, X_r^\xi, \xi_r, \vartheta_r^*)\, dr\right].\end{aligned} \tag{4.46}$$

Due to the admissibility of ξ and the boundedness of the cost coefficients, we can obtain that the term

$$\mathbb{E}_{Q^*}\left[\int_t^s c(Y_r^{t,y}, X_r^\xi, \xi_r)\, dr\right].$$

is finite as s goes to T. By Lemma 4.4.2 and the monotone convergence theorem, letting $s \to T$ in (4.46) yields

$$v(t,y)|x|^p \leq \tilde{J}(t,y,x;\xi,\vartheta^*).$$

Finally note that the equality holds in (4.46) if $\xi = \xi^*$. This yields

$$\begin{aligned} v(t,y)|x|^p &= \mathbb{E}_{Q^*}[v(s,Y_s^{t,y})|X_s^{\xi^*}|^p] + \mathbb{E}_{Q^*}\left[\int_t^s C(Y_r^{t,y},X_r^{\xi^*},\xi_r^*,\vartheta_r^*)\,dr\right] \\ &\to \tilde{J}(t,y,x;\xi^*,\vartheta^*) \quad \text{as } s\to T. \end{aligned}$$

Thus,

$$v(t,y)|x|^p = \tilde{J}(t,y,x;\xi^*,\vartheta^*) \leq \tilde{J}(t,y,x;\xi,\vartheta^*).$$

Step 2. $\tilde{J}(t,y,x;\xi^*,\vartheta) \leq \tilde{J}(t,y,x;\xi^*,\vartheta^*)$ for every ϑ. Let us introduce the sequence of stopping times

$$\tau_n := \inf\{r \in [t,T] : \int_t^r |\vartheta_\rho|^2 d\rho > n\}.$$

Put $\vartheta_r^n = \vartheta_r \mathbb{I}_{r\leq\tau_n}$ and define $W_r^n = W_r + \int_t^r \vartheta_r^n dr$. From the definition of τ_n, it follows that

$$\int_t^T |\vartheta_r^n|^2 dr = \int_t^{\tau_n} |\vartheta_r|^2 dr \leq n. \tag{4.47}$$

Defining $\pi_s^n = \mathcal{E}(\int_t^s \vartheta_r^n dW_r)$, the Novikov condition thus implies that $E[\pi_T^n] = 1$. Setting $dQ^n = \pi_T^n d\mathbb{P}$, by the Girsanov theorem W^n is a Brownian motion under Q. Moreover, $\mathbb{E}[(\pi_s^n)^k] < +\infty$ for every $k > 1$.

As discussed before, we can show that the stochastic integrals $\int_t^s V_r^{t,y}|X_r^{\xi^*}|^p\, dW_r^{\vartheta^n}$ are Q^n-martingales for any $n \in \mathbb{R}$. Hence, we have

$$\begin{aligned} U_t^{t,y}|x|^p &= \mathbb{E}_{Q^n}\left[U_s^{t,y}|X_s^{\xi^*}|^p\right] + \mathbb{E}_{Q^n}\left[\int_t^s C(Y_r^{t,y},X_r^{\xi^*},\xi_r^*,\vartheta_r^n)\,dr\right] \\ &\quad + \mathbb{E}_{Q^n}\left[\int_t^s \left\{\left(\theta^\alpha |Z_r^{t,y}|^{1+\alpha} - \vartheta_r^n Z_r^{t,y} + \frac{1}{\theta}h(\vartheta_r^n)\right)|X_r^{\xi^*}|^p\right\} dr\right] \\ &\geq \mathbb{E}_{Q^n}\left[U_s^{t,y}|X_s^{\xi^*}|^p\right] + \mathbb{E}_{Q^n}\left[\int_t^s C(Y_r^{t,y},X_r^{\xi^*},\xi_r^*,\vartheta_r^n)\,dr\right]. \end{aligned} \tag{4.48}$$

Letting $s \to T$ we get

$$U_t^{t,y}|x|^p \geq \mathbb{E}_{Q^n}\left[\int_t^T C(Y_r^{t,y},X_r^{\xi^*},\xi_r^*,\vartheta_r^n)\,dr\right]$$

by Lemma 4.4.2. We can assume w.l.o.g. that $\mathbb{E}_Q\left[\int_t^T \frac{1}{\theta}h(\vartheta_r)|X_r^{\xi^*}|^p\,dr\right]$ is finite. Since for $r \in [t,T]$,

$$\mathbb{E}\left[\pi_r^n h(\vartheta_r^n)|X_r^{\xi^*}|^p\right] \geq \mathbb{E}\left[\mathbb{E}\left[\pi_r^n h(\vartheta_r^{n-1})|X_r^{\xi^*}|^p|\mathcal{F}_{r\wedge\tau_{n-1}}\right]\right]$$

$$\geq \mathbb{E}\left[\mathbb{E}\left[\pi_r^n|\mathcal{F}_{r\wedge\tau_{n-1}}\right] h(\vartheta_r^{n-1})|X_r^{\xi^*}|^p\right]$$
$$= \mathbb{E}\left[\pi_r^{n-1} h(\vartheta_r^{n-1})|X_r^{\xi^*}|^p\right].$$

the monotone convergence theorem yields,

$$\mathbb{E}_{Q^n}\left[\int_t^T \frac{1}{\theta} h(\vartheta_r^n)|X_r^{\xi^*}|^p\,dr\right] \to \mathbb{E}_Q\left[\int_t^T \frac{1}{\theta} h(\vartheta_r)|X_r^{\xi^*}|^p\,dr\right]$$

as $\tau_n \to \infty, Q$-*a.s.*. Hence,

$$U_t^{t,y}|x|^p \geq \mathbb{E}_Q\left[\int_t^T C(Y_r^{t,y}, X_r^{\xi^*}, \xi_r^*, \vartheta_r)\,dr\right]$$

Recalling that $\tilde{J}(t,y,x;\xi^*,\vartheta^*) = v(t,y)|x|^p$, we have that

$$\tilde{J}(t,y,x;\xi^*,\vartheta) \leq \tilde{J}(t,y,x;\xi^*,\vartheta^*).$$

□

Remark 4.4.3. It was shown that (ξ^*,ϑ^*) is a saddle point of the functional $\tilde{J}$, thus (ξ^*,ϑ^*) is indeed a solution of the robust control problem (4.5). However, $\tilde{J}$ is not convex in ξ for fixed ϑ. So the saddle point (ξ^*,ϑ^*) may not be unique.

4.5. Asymptotic analysis

In this section, we give the proof of Theorem 4.1.8. The main idea is to construct a super- and subsolution to (4.8) by an asymptotic expansion around the benchmark solution and then to apply the comparison principle [Lemma A.3.4].

The following lemma extends the results in [GHS18, Theorem 2.9]. The proof is similar to Section 4.3.

Lemma 4.5.1. *Suppose that $\beta > 2\alpha$ and that Assumptions (L.1)-(L.4) and (F.1)-(F.3) hold. The terminal value problem* (4.4) *admits a unique nonnegative solution v_0 in $C^{0,1}([0,T^-]\times\mathbb{R}^d)$. The solution satisfies the following estimates:*

$$\frac{\underline{c}}{(T-t)^{1/\beta}} \leq v_0 \leq \frac{C_0}{(T-t)^{1/\beta}}, \quad |Dv_0| \leq \frac{C_0}{(T-t)^{1/\beta}}, \quad (t,y)\in[0,T)\times\mathbb{R}^d,$$

for some constant $C_0 > 0$.

Proof. The existence of a classical solution v_0 to (4.4) along with the stated estimates on v_0 has been proved in [GHS18]; the gradient was not given in [GHS18]. In what follows we analyze the $C^{0,1}$ regularity of v_0 under weaker assumptions. As discussed in [GHS18], we can plug the asymptotic ansatz

$$v_0(T-t,y) = \frac{\eta(y)}{t^{1/\beta}} + \frac{u_0(t,y)}{t^{1+1/\beta}}, \quad u_0(t,y) = O(t^2) \text{ uniformly in } y \text{ as } t\to 0. \tag{4.49}$$

into (4.4) and consider instead the PDE

$$\begin{cases} \partial_t u(t,y) = \mathcal{L}u(t,y) + \tilde{f}(t,y,u(t,y)), & t>0\,, y \in \mathbb{R}^d, \\ u(0,y) = 0, & y \in \mathbb{R}^d. \end{cases} \tag{4.50}$$

where

$$\tilde{f}(t,y,u) := t\mathcal{L}\eta(y) + t^p\lambda(y) - \frac{\eta(y)}{\beta}\sum_{k=2}^{\infty}\binom{\beta+1}{k}\left(\frac{u}{t\eta(y)}\right)^k.$$

We now show that this PDE admits a mild solution in $C^{0,1}([0,\delta_0]\times\mathbb{R}^d)$ for some $\delta_0>0$. To this end we consider, similarly to Section 4.3.1, the space

$$E := \{u \in C_b^{0,1}([0,\delta_0]\times\mathbb{R}^d) : \|u(t,\cdot)\| + \|t^{1/2}Du(t,\cdot)\| = O(t^2) \;\; as\; t\to 0\}$$

endowed with the weighted norm

$$\|u\|_E = \sup_{0<t\le\delta_0,\; y\in\mathbb{R}^d} \|t^{-2}u(t,y)\|$$

and define the operator

$$\Gamma_0[u](t,y) = \int_0^t P_{t-s}[\tilde{f}(s,\cdot,u(s,\cdot))](y)ds.$$

Let $R_0>0$ and $\delta_0 \in (0, \underline{c}/R_0]$. Using arguments given in Section 4.3.1 and [GHS18, Section 4], we see that for every u in the closed ball

$$\overline{B}_E(R_0) := \{u \in E : \|u\|_E \le \underline{c}/\delta_0\},$$

the function $\tilde{f}(\cdot, u(\cdot))$ belongs to $C_b([0,\delta_0]\times\mathbb{R}^d)$. In particular, the map Γ_0 is well defined on $\overline{B}_E(R_0)$. Moreover, there exists a constant $L_0>0$ independent of δ_0 such that for all $u,v \in \bar{B}_E(R_0), (t,y)\in[0,\delta_0]\times\mathbb{R}^d$,

$$|\tilde{f}(t,y,u(t,y)) - \tilde{f}(t,y,v(t,y))| \le L_0|u(t,y)-v(t,y)|. \tag{4.51}$$

Now we are ready to carry out the fixed point argument.

Recall that $B(a,b) := \int_0^1 r^{a-1}(1-r)^{b-1}dr$ denotes the Beta function with $a,b>0$. We choose

$$R_0 = 2(1+M\tilde{B}_0)\left(\|\mathcal{L}\eta\| + \|\lambda\|\right),$$

and

$$\delta_0 = \min\{\underline{c}/R_0, 1/\big(2L_0(1+M\tilde{B}_1)\big), 1\},$$

where $L_0>0$ is the Lipschitz constant in (4.51) and $\tilde{B}_0 := B(2,\frac{1}{2}), \tilde{B}_1 := B(3,\frac{1}{2})$. Let $u,v \in \overline{B}_E(R_0)$. For $(t,y)\in[0,\delta_0]\times\mathbb{R}^d$,

$$\begin{aligned} |\Gamma_0[u](t,y) - \Gamma_0[v](t,y)| &= \left|\int_0^t P_{t-s}[\tilde{f}(s,\cdot,u(s,\cdot)) - \tilde{f}(s,\cdot,v(s,\cdot))](y)ds\right| \\ &\le \int_0^t \left\|\tilde{f}(s,\cdot,u(s,\cdot)) - \tilde{f}(s,\cdot,v(s,\cdot))\right\| ds \\ &\le \int_0^t L_0 \left\|u(s,\cdot) - v(s,\cdot)\right\| ds \\ &\le \delta_0 L_0 t^2 \left\|u-v\right\|_E. \end{aligned}$$

Similarly,

$$\begin{aligned}|D\Gamma_0[u](t,y) - D\Gamma_0[v](t,y)| &= \left|\int_0^t DP_{t-s}[\tilde f(s,\cdot,u(s,\cdot)) - \tilde f(s,\cdot,v(s,\cdot))](y)ds\right| \\ &\le M\int_0^t \frac{1}{(t-s)^{1/2}}\left\|\tilde f(s,\cdot,u(s,\cdot)) - \tilde f(s,\cdot,v(s,\cdot))\right\| ds \\ &\le \int_0^t ML_0\frac{1}{(t-s)^{1/2}}\left(s^2\|u-v\|_E\right)ds \\ &\le \delta_0 t^{3/2} ML_0\tilde B_1|u-v\|_E.\end{aligned}$$

Hence

$$\|\Gamma_0[u]-\Gamma_0[v]\|_E \le \frac{1}{2}\|u-v\|_E.$$

To show that Γ_0 maps $\overline{B}_E(R_0)$ into itself, note that $\delta_0 \le 1$ implies $s^k \le 1$ for all $k>0$ and $s\in[0,\delta_0]$. Hence, for every $t\in[0,\delta_0]$

$$\begin{aligned}|\Gamma_0[0](t,y)| &= \left|\int_0^t P_{t-s}[\tilde f(s,\cdot,0)](y)ds\right| \\ &\le \int_0^t \|s\mathcal{L}\eta + s^p\lambda\|\, ds \\ &\le t^2(\|\mathcal{L}\eta\| + \|\lambda\|)\end{aligned}$$

and

$$\begin{aligned}|D\Gamma_0[0](t,y)| &= \left|\int_0^t DP_{t-s}[\tilde f(s,\cdot,0)](y)ds\right| \\ &\le \int_0^t \frac{1}{(t-s)^{1/2}} M\|s\mathcal{L}\eta + s^p\lambda\|\, ds \\ &\le t^{3/2}M\tilde B_0(\|\mathcal{L}\eta\| + \|\lambda\|).\end{aligned}$$

Thus,

$$\|\Gamma_0[u]\|_E \le \|\Gamma_0[u]-\Gamma_0[0]\|_E + \|\Gamma_0[0]\|_E \le R_0.$$

The operator Γ_0 is therefore a contraction from $\overline{B}_E(R_0)$ to itself. Hence, it has a unique fixed point u_0 in $\overline{B}_E(R_0)$. We conclude that Equation (4.50) admits a mild solution in $C_b^{0,1}([0,\delta_0]\times\mathbb{R}^d)$.

In view of the ansatz (4.49), v_0 is a solution to (4.4) in $C_b^{0,1}([T-\delta_0,T^-]\times\mathbb{R}^d)$ and there exists a constant $C>0$ such that for $(t,y)\in[T-\delta_0,T)\times\mathbb{R}^d$,

$$|Dv_0| \le \frac{C}{(T-t)^{1/\beta}}.$$

To establish an *a priori* estimate of Dv_0 on $[0,T-\delta_0]\times\mathbb{R}^d$, we introduce the corresponding FBSDE system

$$\begin{cases} dY_s^{t,y} = b(Y_s^{t,y})ds + \sigma(Y_s^{t,y})dW_s, & s\in[t,T-\delta_0], \\ d\tilde U_s^{t,y} = -F(Y_s^{t,y},\tilde U_s^{t,y})ds + \tilde Z_s^{t,y}dW_s, & s\in[t,T-\delta_0], \\ Y_t^{t,y} = y, \quad \tilde U_{T-\delta_0}^{t,y} = v_0(T-\delta_0, Y_{T-\delta_0}^{t,y}). & \end{cases}$$

By [KPQ97, Theorem 4.1], for $0 \le t \le r \le T - \delta_0$, the map $y \mapsto \tilde{U}_t^{t,y} = v_0(t,y)$ is differentiable and $\tilde{Z}_r^{t,y} = Dv_0(r, Y_r^{t,y})\sigma(Y_r^{t,y})$. The boundedness of Dv_0 can be obtained by the classical BSDE estimates. To conclude, for a constant $C_0 > 0$,

$$|Dv_0| \le \frac{C_0}{(T-t)^{1/\beta}}, \quad (t,y) \in [0,T) \times \mathbb{R}^d. \tag{4.52}$$

□

The next lemma establishes the existence of a unique nonnegative solution to the terminal value problem (4.12) and provides *a priori* estimates on the solution and its derivative.

Lemma 4.5.2. *Suppose that $\beta > 2\alpha$ and that Assumptions (L.1)-(L.4) and (F.1)-(F.3) hold. The terminal value problem* (4.12) *admits a unique nonnegative viscosity solution w_1 in $C^{0,1}([0,T] \times \mathbb{R}^d)$. Moreover, the following estimates hold:*

$$0 \le w_1 \le C_1(T-t)^{1-\alpha/\beta}, \quad |Dw_1| \le C_1(T-t)^{1/2-\alpha/\beta}, \ (t,y) \in [0,T) \times \mathbb{R}^d,$$

for some constant $C_1 > 0$.

Proof. Set $A := |\sigma^* Dv_0|^{1+\alpha}$ and $B := \frac{(\beta+1)v_0^\beta}{\beta\eta^\beta}$. Let $\delta_0 := 1/\|\frac{\mathcal{L}\eta}{\eta}\|$. Using similar arguments to Proposition 3.2.5, we obtain that for $(t,y) \in [T-\delta_0, T) \times \mathbb{R}^d$,

$$\frac{v_0(t,y)^\beta}{\eta(y)^\beta} \ge \frac{\left(1 - \|\frac{\mathcal{L}\eta}{\eta}\|(T-t)\right)^\beta}{T-t}.$$

Hence, there exists $\delta \in (0, \delta_0)$ such that

$$B(t,y) = \frac{(\beta+1)v_0(t,y)^\beta}{\beta\eta(y)^\beta} \ge \frac{1+\beta/2}{\beta(T-t)}, \quad (t,y) \in [T-\delta, T) \times \mathbb{R}^d, \tag{4.53}$$

and so

$$-B(t,y) + \frac{1}{\beta(T-t)} \le \frac{1}{\beta\delta}\mathbb{I}_{t\in[0,T-\delta]} - \frac{1}{2(T-t)}\mathbb{I}_{t\in[T-\delta,T)} \le \frac{1}{\beta\delta} \quad (t,y) \in [0,T) \times \mathbb{R}^d. \tag{4.54}$$

Using the estimates on Dv_0 in Lemma 4.5.1 along with the fact that $\beta > 2\alpha$, we have that

$$\mathbb{E}\left[\int_0^T \left(A(s, Y_s^{t,y})(T-s)^{1/\beta}\right)^2 ds\right] \le \int_0^T \frac{C}{(T-s)^{2\alpha/\beta}}\, ds < +\infty. \tag{4.55}$$

By (4.54) and (4.55), it follows from the Feynman-Kac formula [Par99, Theorem 3.2] that

$$w_1(t,y) := \mathbb{E}\left[\int_t^T \exp\left(\int_t^s \left(-B(r, Y_r^{t,y}) + \frac{1}{\beta(T-r)}\right) dr\right) A(s, Y_s^{t,y})(T-s)^{1/\beta}\, ds\right]$$

is the unique viscosity solution to the terminal value problem (4.12) on $[0,T]\times\mathbb{R}^d$. Moreover, we have for $(t,y)\in[0,T)\times\mathbb{R}^d$ that

$$\begin{aligned} w_1(t,y) \le & \mathbb{E}\left[\int_t^T \exp\left(\int_t^s \frac{1}{\beta\delta}\,dr\right) A(s,Y_s^{t,y})(T-s)^{1/\beta}\,ds\right] \\ \le & \int_t^T e^{T/(\beta\delta)}\frac{C}{(T-s)^{\alpha/\beta}}\,ds \\ \le & C_1(T-t)^{1-\alpha/\beta} \end{aligned} \tag{4.56}$$

for some constant C_1.

Next, we study the derivative of w_1. For any $\varepsilon\in(0,T)$, restricting the PDE (4.12) to $[0,T-\varepsilon]$,

$$\begin{cases} -\partial_t v(t,y) - \mathcal{L}v(t,y) - f_1(t,y,v(t,y)) = 0, & (t,y)\in[0,T-\varepsilon)\times\mathbb{R}^d, \\ v(T-\varepsilon,y) = w_1(T-\varepsilon,y) & y\in\mathbb{R}^d, \end{cases}$$

Since A,B are bounded on $[0,T-\varepsilon]$, it follows from the Bismut-Elworthy formula [FT02, Theorem 4.2] that $w_1(t,\cdot)$ is differentiable for $t\in[0,T-\varepsilon]$ and

$$\begin{aligned} |Dw_1(t,y)| \le & \frac{C}{(T-\varepsilon-t)^{1/2}}\|w_1(T-\varepsilon,\cdot)\| + \int_t^{T-\varepsilon}\frac{C}{(s-t)^{1/2}}\Bigg((T-s)^{1/\beta}\|A(s,\cdot)\| \\ & + \left(\|B(s,\cdot)\| + \frac{1}{\beta(T-s)}\right)\|w_1(s,\cdot)\|\Bigg)\,ds \end{aligned}$$

for $(t,y)\in[0,T-\varepsilon)\times\mathbb{R}^d$. Using the estimates on v_0, w_1 we get that

$$\begin{aligned} |Dw_1(t,y)| &\le \frac{C}{(T-\varepsilon-t)^{1/2}}\varepsilon^{1-\alpha/\beta} + C\int_t^T \frac{1}{(s-t)^{1/2}}(T-s)^{-\alpha/\beta}\,ds \\ &\le \frac{C}{(T-\varepsilon-t)^{1/2}}(T-t)^{1-\alpha/\beta}, \quad (t,y)\in[0,T-\varepsilon)\times\mathbb{R}^d, \end{aligned}$$

where C is independent of ε. By letting ε go to zero, we see that (by an adjustment of C_1 if necessary)

$$|Dw_1(t,y)| \le C_1(T-t)^{1/2-\alpha/\beta}, \quad (t,y)\in[0,T)\times\mathbb{R}^d. \tag{4.57}$$

□

By the transformation $v_1 = \frac{1}{(T-t)^{1/\beta}}w_1$, we know that v_1 is a solution to the equation

$$-\partial_t v(t,y) - \mathcal{L}v(t,y) - |\sigma Dv_0|^{1+\alpha} + \frac{(\beta+1)v_0^\beta}{\beta\eta^\beta}v = 0, \quad (t,y)\in[0,T)\times\mathbb{R}. \tag{4.58}$$

Since $\beta>2\alpha$, there exists a positive constant C_2 such that for all $(t,y)\in[0,T)\times\mathbb{R}^d$,

$$\begin{aligned} 0 \le v_1 \le C_1(T-t)^{1-(1+\alpha)/\beta} \le C_2(T-t)^{-1/\beta}, \\ |Dv_1| \le C_1(T-t)^{1/2-(1+\alpha)/\beta} \le C_2(T-t)^{-1/\beta}. \end{aligned} \tag{4.59}$$

Armed with these estimates, we are now ready to prove the asymptotic result.

Proof of Theorem 4.1.8. Let δ be as in (4.53) and set $b := \frac{\bar{C}^\beta}{(\beta+1)\underline{c}^\beta \delta^{1/\beta}}$. Our goal is to find two constants $L_1 > 0, L_2 < 0$ such that

$$u_i = v_0 + \theta^\alpha v_1 + \theta^{2\alpha} L_i \left(b + \frac{1}{(T-t)^{1/\beta}} \right), \quad i = 1,2$$

is a supersolution (i=1), respectively a subsolution (i=2) to (4.8). For $i = 1,2$,

$$\begin{aligned}
&- \theta^\alpha |\sigma D u_i|^{1+\alpha} + \frac{u_i^{\beta+1}}{\beta \eta^\beta} - \lambda(y) \\
&= - \theta^\alpha |\sigma (D v_0 + \theta^\alpha D v_1)|^{1+\alpha} + \frac{\left(v_0 + \theta^\alpha v_1 + \theta^{2\alpha} L_i (b + \frac{1}{(T-t)^{1/\beta}}) \right)^{\beta+1}}{\beta \eta^\beta} - \lambda(y) \\
&= - \theta^\alpha |\sigma D v_0|^{1+\alpha} + \frac{v_0^{\beta+1} + (\beta+1)\theta^\alpha v_0^\beta v_1}{\beta \eta^\beta} - \lambda(y) + \theta^{2\alpha} \frac{L_i}{\beta (T-t)^{1/\beta+1}} + \mathcal{I}_i,
\end{aligned}$$

where $\mathcal{I}_i := \mathcal{I}_i^0 + \mathcal{I}_i^1 + \mathcal{I}_i^2$ and $\mathcal{I}_i^0, \mathcal{I}_i^1, \mathcal{I}_i^2$ are given by

$$\begin{aligned}
\mathcal{I}_i^0 &:= -\theta^{2\alpha} L_i \frac{1}{\beta (T-t)^{1/\beta+1}}; \\
\mathcal{I}_i^1 &:= \frac{\left(v_0 + \theta^\alpha v_1 + \theta^{2\alpha} L_i (b + \frac{1}{(T-t)^{1/\beta}}) \right)^{\beta+1}}{\beta \eta^\beta} - \frac{v_0^{\beta+1} + (\beta+1)\theta^\alpha v_0^\beta v_1}{\beta \eta^\beta}; \\
\mathcal{I}_i^2 &:= \theta^\alpha |\sigma D v_0|^{1+\alpha} - \theta^\alpha |\sigma (D v_0 + \theta^\alpha D v_1)|^{1+\alpha}.
\end{aligned}$$

It is sufficient to prove that $\mathcal{I}_1 > 0$ (supersolution) and that $\mathcal{I}_2 < 0$ (subsolution) on $[0,T) \times \mathbb{R}^d$.

The second order Taylor approximation around v_0 in the first summand of $\mathcal{I}_i^1$ yields a function ζ satisfying $\min\{v_0, u_i\} \leq \zeta \leq \max\{v_0, u_i\}$ such that

$$\begin{aligned}
\mathcal{I}_i^1 =& \theta^{2\alpha} L_i \frac{1}{\beta \eta^\beta} (\beta+1) v_0^\beta (b + \frac{1}{(T-t)^{1/\beta}}) \\
&+ \frac{1}{2\eta^\beta} (\beta+1) \zeta^{\beta-1} \left(\theta^\alpha v_1 + \theta^{2\alpha} L_i (b + \frac{1}{(T-t)^{1/\beta}}) \right)^2.
\end{aligned}$$

The mean value theorem along with the triangle inequality also yields a constant $\tilde{C}_0 > 0$ such that

$$\begin{aligned}
|\mathcal{I}_i^2| &\leq \theta^{2\alpha} \bar{C}^\alpha (|D v_0|^\alpha + |D v_0 + \theta^\alpha D v_1|^\alpha) |D v_1| \\
&\leq \theta^{2\alpha} \tilde{C}_0 (T-t)^{(1+\alpha)/\beta} \\
&\leq \theta^{2\alpha} \frac{\tilde{C}_0 T^{1-\alpha/\beta}}{(T-t)^{1/\beta+1}}.
\end{aligned}$$

Step 1: Construction of supersolution. Using the lower bound of v_0 in Lemma 4.5.1, we have that

$$\frac{\eta(y)^\beta}{(\beta+1) v_0(t,y)^\beta (T-t)^{1/\beta+1}} \leq \frac{\bar{C}^\beta}{(\beta+1)\underline{c}^\beta (T-t)^{1/\beta}} \leq \frac{\bar{C}^\beta}{(\beta+1)\underline{c}^\beta \delta^{1/\beta}} = b.$$

Set $c := \min\{\frac{1}{2}, \frac{(\beta+1)\underline{c}^\beta}{\beta \tilde{C}^\beta}\}$. The preceding inequality along with the inequality (4.53) yields

$$-\frac{1}{\beta(T-t)^{1/\beta+1}} + \frac{1}{\beta\eta^\beta}(\beta+1)v_0^\beta\left(b + \frac{1}{(T-t)^{1/\beta}}\right) \geq c\frac{1}{(T-t)^{1/\beta+1}}. \tag{4.60}$$

Since the second term in the definition of $\mathcal{I}_1^1$ is nonnegative, we have that

$$\mathcal{I}_1 \geq c\theta^{2\alpha}\frac{L_1}{(T-t)^{1/\beta+1}} - \theta^{2\alpha}\frac{\tilde{C}_0 T^{1-\alpha/\beta}}{(T-t)^{1/\beta+1}}.$$

Choosing $L_1 > \frac{\tilde{C}_0 T^{1-\alpha/\beta}}{c}$, we obtain that $\mathcal{I}_1 > 0$.

Step 2: Construction of subsolution. Using the lower bound of v_0 in Lemma 4.5.1 again and choosing $L_2 < 0, \theta > 0$ such that $\theta^{2\alpha}|L_2|(T^{1/\beta}b+1) \leq \frac{\underline{c}}{2}$, we obtain that $u_2 \geq \frac{\underline{c}}{2(T-t)^{1/\beta}} \geq 0$. Different from Step 1, an additional estimate on the second term in the definition of $\mathcal{I}_2^1$ is needed to obtain that $\mathcal{I}_2 < 0$. Since $\min\{v_0, u_2\} \leq \zeta \leq \max\{v_0, u_2\}$, we see that $\zeta(T-t)^{1/\beta}$ can be bounded both from below and above. Therefore, there exists a constant $\tilde{C}_1 > 0$ such that

$$\frac{1}{2\eta^\beta}(\beta+1)\zeta^{\beta-1}\left(\theta^\alpha v_1 + \theta^{2\alpha}L_i(b + \frac{1}{(T-t)^{1/\beta}})\right)^2 \leq \theta^{2\alpha}\frac{\tilde{C}_1}{(T-t)^{1/\beta+1}}.$$

By the inequality (4.60) and the nonpositivity of L_2, we have that

$$-\frac{L_2}{\beta(T-t)^{1/\beta+1}} + \frac{1}{\beta\eta^\beta}(\beta+1)v_0^\beta L_2(b + \frac{1}{(T-t)^{1/\beta}}) \leq c\frac{L_2}{(T-t)^{1/\beta+1}}.$$

Thus,

$$\mathcal{I}_2 \leq c\theta^{2\alpha}\frac{L_2}{(T-t)^{1/\beta+1}} + \theta^{2\alpha}\frac{\tilde{C}_0 T^{1-\alpha/\beta}}{(T-t)^{1/\beta+1}} + \theta^{2\alpha}\frac{\tilde{C}_1}{(T-t)^{1/\beta+1}} < 0$$

if we first choose

$$L_2 < -\frac{\tilde{C}_1 + \tilde{C}_0 T^{1-\alpha/\beta}}{c}$$

and then

$$\theta < \min\{1, \sqrt[2\alpha]{\underline{c}/(2|L_2|(T^{1/\beta}b+1))}\}.$$

Hence u_2 is a nonnegative viscosity subsolution to (4.8). By Lemma A.3.4, we then have that $u_2 \leq v \leq u_1$. Thus, the desired equality (4.11) follows from

$$\theta^\alpha w_1 + \theta^{2\alpha}L_2(b(T-t)^{1/\beta}+1) \leq w - w_0 \leq \theta^\alpha w_1 + \theta^{2\alpha}L_1(b(T-t)^{1/\beta}+1).$$

□

A. Appendix

A.1. Legend labels in Figures 2.1-2.5

Denote $X_i^*, i = 1, 2$, the ith asset's optimal position, and $\xi_i^*, i = 1, 2$, the ith asset's trading strategy. All solid lines in Figures are for $i = 1$, all dashed lines for $i = 2$.

Table A.1.: The legend labels in Figures 2.1 and 2.2.

Figure / Line	Figure 2.1:right side ξ_i^*	Figure 2.2: left side X_i^*, right side ξ_i^*
red	$k = 1$	$\lambda_1 = 10, k = 1$
blue	$k = 0.5$	$\lambda_1 = 0.5, k = 1$
green	$k = 0$	$\lambda_1 = 0.1, k = 1$
yellow	$k = -0.5$	$\lambda_1 = 0.1, k = -1$
cyan	$k = -1$	$\lambda_1 = 0.5, k = -1$
black		$\lambda_1 = 10, k = -1$

Table A.2.: The legend labels in Figure 2.4.

Figure / Line	Figure 2.4: left side X_i^*, right side ξ_i^*
red	$\gamma_1 = 100, k = 0.5$
blue	$\gamma_1 = 10, k = 0.5$
green	$\gamma_1 = 0.1, k = 0.5$
yellow	$\gamma_1 = 0.1, k = -0.5$
cyan	$\gamma_1 = 10, k = -0.5$
black	$\gamma_1 = 100, k = -0.5$

Table A.3.: The legend labels in Figure 2.5.

Figure / Line	Figure 2.5: left side X_i^*, right side ξ_i^*
red	$\rho_1 = 100, k = 1$
blue	$\rho_1 = 10, k = 1$
green	$\rho_1 = 0.1, k = 1$
yellow	$\rho_1 = 0.1, k = -1$
cyan	$\rho_1 = 10, k = -1$
black	$\rho_1 = 100, k = -1$

A.2. Some estimates

Lemma A.2.1. *Let $n > n_0$ for n_0 as in* (2.21)*. Recall that p^n, q^n are defined by* (2.29) *and* (2.31)*. For fixed $t \in [0,T)$, there exists a constant L independent of n, s such that*

$$\begin{aligned} e^{\int_t^s p^n(u)du} &\leq L\mathbb{I}_{s\in[0,T_0)} + \frac{L}{T-s+\frac{\lambda_{\min}}{n-\gamma_{\min}-\lambda_{\min}(1+\sqrt{1+\alpha})}}\mathbb{I}_{s\in[T_0,T]}; \\ e^{\int_t^s -q^n(u)du} &\leq L\mathbb{I}_{s\in[0,T_0)} + L\left(T-s+\frac{\lambda_{\max}}{n-\gamma_{\max}+\lambda_{\max}}\right)\mathbb{I}_{s\in[T_0,T]}, \end{aligned} \tag{A.1}$$

with

$$\alpha = \frac{\|\Sigma\|_{L^\infty} + 2\gamma_{\max}\|\rho\|_{L^\infty}}{\lambda_{\min}}.$$

Proof. We first introduce simpler bounds $\tilde{p}^n, \tilde{q}^n$ for p^n, q^n to simplify the calculations (cf. [Kra11, Corollary 2.2.3]). For $n > n_0$, $\tilde{q}^n \leq q^n, p^n \leq \tilde{p}^n$ where $\tilde{q}^n, \tilde{p}^n$are given by

$$\tilde{p}^n(t) = \begin{cases} p^n(t), & t \in [0,T_0); \\ \dfrac{1}{T-t+\frac{1}{\frac{n-\gamma_{\min}}{\lambda_{\min}}-\sqrt{1+\alpha}-1}} + 1 + \sqrt{1+\alpha}, & t \in [T_0,T], \end{cases}$$

$$\tilde{q}^n(t) = \begin{cases} q^n(t), & t \in [0,T_0); \\ \dfrac{1}{T-t+\frac{1}{\frac{n-\gamma_{\max}}{\lambda_{\max}}+1}} - 1, & t \in [T_0,T]. \end{cases}$$

Hence we need to prove that,

$$\begin{aligned} e^{\int_t^s \tilde{p}^n(u)du} &\leq L\mathbb{I}_{s\in[0,T_0)} + \frac{L}{T-s+\frac{\lambda_{\min}}{n-\gamma_{\min}-\lambda_{\min}(1+\sqrt{1+\alpha})}}\mathbb{I}_{s\in[T_0,T]}; \\ e^{\int_t^s -\tilde{q}^n(u)du} &\leq L\mathbb{I}_{s\in[0,T_0)} + L\left(T-s+\frac{\lambda_{\max}}{n-\gamma_{\max}+\lambda_{\max}}\right)\mathbb{I}_{s\in[T_0,T]}. \end{aligned}$$

For $0 \leq t < s < T_0$,

$$\begin{aligned} e^{\int_t^s \tilde{p}^n(u)du} &= e^{\tilde{p}^n(0)(s-t)} \leq e^{\tilde{p}^n(0)T_0}; \\ e^{\int_t^s -\tilde{q}^n(u)du} &= e^{-\tilde{q}^n(0)(s-t)} \leq 1. \end{aligned}$$

For $0 \leq t < T_0 \leq s \leq T$,

$$\begin{aligned} e^{\int_t^s \tilde{p}^n(u)du} &= e^{\tilde{p}^n(0)(T_0-t)}e^{(1+\sqrt{1+\alpha})(s-T_0)}\left(\frac{T-T_0+\frac{\lambda_{\min}}{n-\gamma_{\min}-\lambda_{\min}(1+\sqrt{1+\alpha})}}{T-s+\frac{\lambda_{\min}}{n-\gamma_{\min}-\lambda_{\min}(1+\sqrt{1+\alpha})}}\right) \\ &\leq e^{\tilde{p}^n(0)T_0}e^{(1+\sqrt{1+\alpha})T}\left(\frac{T-T_0+\frac{\lambda_{\min}}{n_2-\gamma_{\min}-\lambda_{\min}(1+\sqrt{1+\alpha})}}{T-s+\frac{\lambda_{\min}}{n-\gamma_{\min}-\lambda_{\min}(1+\sqrt{1+\alpha})}}\right) \end{aligned}$$

and

$$\begin{aligned}
e^{\int_t^s -\tilde{q}^n(u)du} &= e^{-\tilde{q}^n(0)(T_0-t)} e^{(s-T_0)} \left(\frac{T-s+\frac{\lambda_{\max}}{n-\gamma_{\max}+\lambda_{\max}}}{T-T_0+\frac{\lambda_{\max}}{n-\gamma_{\max}+\lambda_{\max}}} \right) \\
&\le e^T \left(\frac{T-s+\frac{\lambda_{\max}}{n-\gamma_{\max}+\lambda_{\max}}}{T-T_0} \right).
\end{aligned}$$

For $T_0 \le t < s \le T$,

$$\begin{aligned}
e^{\int_t^s \tilde{p}^n(u)du} &= e^{(1+\sqrt{1+\alpha})(s-t)} \left(\frac{T-t+\frac{\lambda_{\min}}{n-\gamma_{\min}-\lambda_{\min}(1+\sqrt{1+\alpha})}}{T-s+\frac{\lambda_{\min}}{n-\gamma_{\min}-\lambda_{\min}(1+\sqrt{1+\alpha})}} \right) \\
&\le e^{(1+\sqrt{1+\alpha})T} \left(\frac{T-t+\frac{\lambda_{\min}}{n_2-\gamma_{\min}-\lambda_{\min}(1+\sqrt{1+\alpha})}}{T-s+\frac{\lambda_{\min}}{n-\gamma_{\min}-\lambda_{\min}(1+\sqrt{1+\alpha})}} \right)
\end{aligned}$$

and

$$\begin{aligned}
e^{\int_t^s -\tilde{q}^n(u)du} &= e^{(s-t)} \left(\frac{T-s+\frac{\lambda_{\max}}{n-\gamma_{\max}+\lambda_{\max}}}{T-t+\frac{\lambda_{\max}}{n-\gamma_{\max}+\lambda_{\max}}} \right) \\
&\le e^T \left(\frac{T-s+\frac{\lambda_{\max}}{n-\gamma_{\max}+\lambda_{\max}}}{T-t} \right).
\end{aligned}$$

Therefore, for fixed $t \in [0,T)$, there exists a constant L independent of n, s such that

$$\begin{aligned}
e^{\int_t^s p^n(u)du} &\le L\mathbb{I}_{s\in[0,T_0)} + \frac{L}{T-s+\frac{\lambda_{\min}}{n-\gamma_{\min}-\lambda_{\min}(1+\sqrt{1+\alpha})}} \mathbb{I}_{s\in[T_0,T]}; \\
e^{\int_t^s -q^n(u)du} &\le L\mathbb{I}_{s\in[0,T_0)} + L\left(T-s+\frac{\lambda_{\max}}{n-\gamma_{\max}+\lambda_{\max}} \right) \mathbb{I}_{s\in[T_0,T]}.
\end{aligned}$$

□

A.3. Some Comparison principles

In this section, we state and prove comparison principles for solutions to PDEs with superlinear gradient term. Both finite and singular terminal values will be considered.

We refer to [LL10] as an important reference for PDEs with superlinear gradient term. The following comparison principle can be seen as a corollary to [LL10, Theorem 3.1].

Proposition A.3.1. *Assume that (L.1)-(L.3) hold. Let* $v \in LSC_m([0,T] \times \mathbb{R}^d)$ *and* $u \in USC_m([0,T] \times \mathbb{R}^d)$ *be a nonnegative viscosity super- and a nonnegative viscosity subsolution to the following PDE:*

$$\begin{cases} -\partial_t v(t,y) - \mathcal{L}v(t,y) - H(y, Dv(t,y)) - f(t,y) = 0, & (t,y) \in [0,T) \times \mathbb{R}^d, \\ v(T,y) = \phi(y) & y \in \mathbb{R}^d, \end{cases}$$

If $\phi \in C_m(\mathbb{R}^d), f \in C_m([0,T] \times \mathbb{R}^d)$*, then*

$$u \le v \quad \text{on} \quad [0,T] \times \mathbb{R}^d.$$

Let us now consider the more general PDE

$$\begin{cases} -\partial_t v(t,y) - \mathcal{L}v(t,y) - H(y, Dv(t,y)) - F(y, v(t,y)) = 0, & (t,y) \in [0,T) \times \mathbb{R}^d, \\ v(T,y) = \phi(y), & y \in \mathbb{R}^d. \end{cases} \tag{A.2}$$

A comparison principle for such PDEs is obtained in [LL10] under a Lipschitz continuity assumption of F on v. This condition is not satisfied in our case; we only have monotonicity. Additional assumptions on the solution are thus required to establish a comparison principle. Here, we make a weaker assumption on the coefficients than (F.1) and (F.2).

(F.4) The coefficients $\eta, \lambda, 1/\eta : \mathbb{R}^d \to [0,\infty)$ are continuous and λ is of polynomial growth of order m.

Proposition A.3.2. *Assume that (L.1)-(L.3) and (F.4) hold and that* $\phi \in C_m(\mathbb{R}^d)$*. Let* $v \in LSC([0,T] \times \mathbb{R}^d) \cap \mathcal{SSG}_m^+$ *and* $u \in USC([0,T] \times \mathbb{R}^d) \cap \mathcal{SSG}_m^-$ *be a nonnegative viscosity super- and a nonnegative viscosity subsolution to* (A.2)*. Suppose that there exists* $\hat{C} > 0$ *such that for all* $(t,y) \in [0,T] \times \mathbb{R}^d$,

$$u^{\beta+1}(t,y), v^{\beta+1}(t,y) \le \hat{C}\eta^\beta(y)\langle y\rangle^m. \tag{A.3}$$

Then,

$$u \le v \quad \text{on} \quad [0,T] \times \mathbb{R}^d.$$

Proof. Step 1: linearization. For $\rho \in (0,1)$, it is easy to verify that $\tilde{v} := \rho v$ is a viscosity supersolution of the following PDE:

$$-\partial_t \tilde{v}(t,y) - \mathcal{L}\tilde{v}(t,y) - \rho H(y, \frac{D\tilde{v}(t,y)}{\rho}) - \rho F(y, \frac{\tilde{v}(t,y)}{\rho}) = 0, (t,y) \in [0,T) \times \mathbb{R}^d,$$

In what follows, we show that $w := u - \tilde{v}$ is a viscosity subsolution of the following extremal PDE:

$$-\partial_t v(t,y) - \mathcal{L}v(t,y) - (\frac{1-\rho}{2})^{-\alpha}\bar{C}^{\alpha+1}|Dv|^{\alpha+1} - (1-\rho)\left[\lambda(\bar{y}) + \frac{1+\beta}{\beta}\hat{C}\langle y\rangle^m\right] = 0, \tag{A.4}$$

for $(t,y) \in [0,T) \times \mathbb{R}^d \cap \{w > 0\}$.

Let $\varphi \in C^2([0,T) \times \mathbb{R}^d)$ be a test function and $(\bar{t},\bar{y}) \in [0,T) \times \mathbb{R}^d \cap \{w > 0\}$ be a local maximum of $w - \varphi$. We may assume that this maximum is strict in the set $[\bar{t} - r, \bar{t} + r] \times \bar{B}_r(\bar{y}) \subset [0,T) \times \mathbb{R}^d$ for small $r \in (0,1)$; we choose $[0,r] \times \bar{B}_r(\bar{y})$ if $\bar{t} = 0$. Let

$$\Phi(t,x,y) := \frac{|x-y|^2}{2\varepsilon} + \varphi(t,y)$$

and

$$M_\varepsilon := \max_{t\in[\bar{t}-r,\bar{t}+r], x,y\in\bar{B}_r(\bar{y})} \big(u(t,x) - \tilde{v}(t,y) - \Phi(t,x,y)\big).$$

This maximum is attained at a point $(t_\varepsilon, x_\varepsilon, y_\varepsilon)$ and is strict. We know that

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \to 0 \text{ and } M_\varepsilon \to u(\bar{t},\bar{y}) - \tilde{v}(\bar{t},\bar{y}) - \varphi(\bar{t},\bar{y}) \text{ as } \varepsilon \to 0.$$

We now apply [CIL92, Theorem 8.3]. In terms of their notation we have that $k = 2, u_1 = u, u_2 = -\tilde{v}, \varphi(t,x,y) = \Phi(t,x,y)$. Moreover, we recall the property that $\bar{\mathcal{P}}^{2,-}(\tilde{v}) = -\bar{\mathcal{P}}^{2,+}(-\tilde{v})$. Then, setting $p_\varepsilon = \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}$, we have that

$$\begin{aligned} \partial_x\Phi(t_\varepsilon,x_\varepsilon,y_\varepsilon) &= p_\varepsilon, \\ -\partial_y\Phi(t_\varepsilon,x_\varepsilon,y_\varepsilon) &= p_\varepsilon - D\varphi(t_\varepsilon,y_\varepsilon) \end{aligned}$$

and that

$$A = D^2\Phi(t_\varepsilon,x_\varepsilon,y_\varepsilon) = \begin{pmatrix} \frac{I}{\varepsilon} & -\frac{I}{\varepsilon} \\ -\frac{I}{\varepsilon} & \frac{I}{\varepsilon} + D^2\varphi(t_\varepsilon,y_\varepsilon) \end{pmatrix}.$$

From this we conclude that for every $\iota > 0$, there exist $a_1, a_2 \in \mathbb{R}, X, Y \in \mathcal{S}^d$ such that

$$(a_1, p_\varepsilon, X) \in \bar{\mathcal{P}}^{2,+}u(t_\varepsilon,x_\varepsilon), \quad (a_2, p_\varepsilon - D\varphi(t_\varepsilon,y_\varepsilon), Y) \in \bar{\mathcal{P}}^{2,-}\tilde{v}(t_\varepsilon,y_\varepsilon),$$

such that $a_1 - a_2 = \partial_t\Phi(t_\varepsilon,x_\varepsilon,y_\varepsilon) = \varphi_t(t_\varepsilon,x_\varepsilon)$ and such that

$$-(\frac{1}{\iota} + \|A\|)I \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \iota A^2. \tag{A.5}$$

From the definition of viscosity solution, we obtain that

$$-a_1 - b(x_\varepsilon)p_\varepsilon - \frac{1}{2}\mathrm{tr}\,[\sigma\sigma^*(x_\varepsilon)X] - F(x_\varepsilon,u) \le H(x_\varepsilon,p_\varepsilon)$$

and that

$$
\begin{aligned}
&- a_2 - b(y_\varepsilon)(p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon)) - \frac{1}{2}\mathrm{tr}\,[\sigma\sigma^*(y_\varepsilon)Y] - \rho F(y_\varepsilon, \frac{\tilde{v}}{\rho}) \\
\geq &\rho H(y_\varepsilon, \frac{p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon)}{\rho}).
\end{aligned}
$$

Subtracting the two inequalities, we have

$$
\begin{aligned}
&- \partial_t\varphi_t(t_\varepsilon, y_\varepsilon) + b(y_\varepsilon)(p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon)) - b(x_\varepsilon)p_\varepsilon \\
&+ \frac{1}{2}\mathrm{tr}\,[\sigma\sigma^*(y_\varepsilon)Y] - \frac{1}{2}\mathrm{tr}\,[\sigma\sigma^*(x_\varepsilon)X] \\
&+ \rho F(y_\varepsilon, \frac{\tilde{v}}{\rho}) - F(x_\varepsilon, u) \leq H(x_\varepsilon, p_\varepsilon) - \rho H(y_\varepsilon, \frac{p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon)}{\rho}).
\end{aligned}
$$

We are now going to estimate the terms involving the drift, the volatility, and the functions F and H separately.

- Since b is Lipschitz continuous,

$$
\begin{aligned}
b(y_\varepsilon)(p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon)) - b(x_\varepsilon)p_\varepsilon &= -b(y_\varepsilon)D\varphi(t_\varepsilon, y_\varepsilon) + (b(y_\varepsilon) - b(x_\varepsilon))p_\varepsilon \\
&\geq -b(y_\varepsilon)D\varphi(t_\varepsilon, y_\varepsilon) - \bar{C}\varepsilon^{-1}|x_\varepsilon - y_\varepsilon|^2.
\end{aligned}
$$

- In order to estimate the volatility term we denote by $(e_i)_{1\leq i\leq \tilde{d}}$ the canonical basis of $\mathbb{R}^{\tilde{d}}$. By using (A.5) and the Lipschitz continuity of σ, we obtain

$$
\begin{aligned}
&\mathrm{tr}\,[\sigma\sigma^*(x_\varepsilon)X] - \mathrm{tr}\,[\sigma\sigma^*(y_\varepsilon)Y] \\
=&\sum_{i=1}^{\tilde{d}}\langle X\sigma(x_\varepsilon)e_i, \sigma(x_\varepsilon)e_i\rangle - \sum_{i=1}^{\tilde{d}}\langle Y\sigma(y_\varepsilon)e_i, \sigma(y_\varepsilon)e_i\rangle \\
\leq&\sum_{i=1}^{\tilde{d}}\langle D^2\varphi(t_\varepsilon, y_\varepsilon)\sigma(y_\varepsilon)e_i, \sigma(y_\varepsilon)e_i\rangle + \frac{1}{\varepsilon}|\sigma(x_\varepsilon) - \sigma(y_\varepsilon)|^2 + \omega(\frac{\iota}{\varepsilon^2}) \\
\leq&\mathrm{tr}\left[\sigma\sigma^*(y_\varepsilon)D^2\varphi(t_\varepsilon, y_\varepsilon)\right] + \bar{C}^2\varepsilon^{-1}|x_\varepsilon - y_\varepsilon|^2 + \omega(\frac{\iota}{\varepsilon^2})
\end{aligned}
$$

  where ω is a modulus of continuity which is independent of ι and ε.

- We now estimate $\tilde{F} := \rho F(y_\varepsilon, \frac{\tilde{v}}{\rho}) - F(x_\varepsilon, u)$. To this end, we first observe that

$$
u(t_\varepsilon, x_\varepsilon) - \tilde{v}(t_\varepsilon, y_\varepsilon) - \varphi(t_\varepsilon, y_\varepsilon) \geq M_\varepsilon \geq u(\bar{t}, \bar{y}) - \tilde{v}(\bar{t}, \bar{y}) - \varphi(\bar{t}, \bar{y}).
$$

  Since $(\bar{t}, \bar{y}) \in \{w > 0\}$ and φ is continuous, we can fix r small enough to obtain that

$$
u(t_\varepsilon, x_\varepsilon) - \tilde{v}(t_\varepsilon, y_\varepsilon) \geq 0.
$$

Recalling the definition of F in (3.7), the fact that $F(y,\cdot)$ is decreasing on $\mathbb{R}_+$ and the fact that $\rho(1-\rho^\beta) < (1+\beta)(1-\rho)$ for $0<\rho<1$, this yields

$$\begin{aligned}
\tilde{F} &= \rho F(y_\varepsilon, \frac{\tilde{v}}{\rho}) - F(y_\varepsilon, u) + F(y_\varepsilon, u) - F(x_\varepsilon, u) \\
&\geq (\rho-1)\lambda(y_\varepsilon) + \frac{|u|^{\beta+1}}{\beta\eta(y_\varepsilon)^\beta} - \rho^{-\beta}\frac{|\tilde{v}|^{\beta+1}}{\beta\eta(y_\varepsilon)^\beta} \\
&\quad - \omega_R(|x_\varepsilon - y_\varepsilon|) \\
&= (\rho-1)\lambda(y_\varepsilon) + \frac{|u|^{\beta+1}}{\beta\eta(y_\varepsilon)^\beta} - \frac{|\tilde{v}|^{\beta+1}}{\beta\eta(y_\varepsilon)^\beta} \\
&\quad - \rho(1-\rho^\beta)\frac{|v|^{\beta+1}}{\beta\eta(y_\varepsilon)^\beta} - \omega_R(|x_\varepsilon - y_\varepsilon|) \\
&\geq -(1-\rho)\lambda(y_\varepsilon) - (1+\beta)(1-\rho)\frac{|v|^{\beta+1}}{\beta\eta(y_\varepsilon)^\beta} - \omega_R(|x_\varepsilon - y_\varepsilon|) \\
&\geq -(1-\rho)\left[\lambda(y_\varepsilon) + \frac{1+\beta}{\beta}\hat{C}\langle y_\varepsilon\rangle^m\right] - \omega_R(|x_\varepsilon - y_\varepsilon|)
\end{aligned} \tag{A.6}$$

where ω_R denotes the modulus of continuity with $R := |\bar{y}| + r$.

- We finally estimate $\tilde{H} := H(x_\varepsilon, p_\varepsilon) - \rho H(y_\varepsilon, \frac{p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon)}{\rho})$. By convexity, we have, for $z_1, z_2 \in \mathbb{R}^d$, that

$$|z_1|^{\alpha+1} - \rho|\frac{z_2}{\rho}|^{\alpha+1} \leq (1-\rho)|\frac{z_1 - z_2}{1-\rho}|^{\alpha+1}.$$

Hence,

$$\begin{aligned}
&H(x_\varepsilon, p_\varepsilon) - \rho H(y_\varepsilon, \frac{p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon)}{\rho}) \\
\leq&(1-\rho)\left|\frac{\sigma(x_\varepsilon)p_\varepsilon - \sigma(y_\varepsilon)(p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon))}{1-\rho}\right|^{\alpha+1} \\
\leq&(\frac{1-\rho}{2})^{-\alpha}\bar{C}^{\alpha+1}\left(|D\varphi(t_\varepsilon, y_\varepsilon)|^{\alpha+1} + (|x_\varepsilon - y_\varepsilon|\cdot|p_\varepsilon|)^{\alpha+1}\right)
\end{aligned}$$

where (L.2), (L.3) are used in the last inequality.

Denoting a generic modulus of continuity independent of ι and ε by ω, we thus get

$$\begin{aligned}
&-\partial_t\varphi(t_\varepsilon, y_\varepsilon) - \mathcal{L}\varphi(t_\varepsilon, y_\varepsilon) - (\frac{1-\rho}{2})^{-\alpha}\bar{C}^{\alpha+1}|D\varphi(t_\varepsilon, y_\varepsilon)|^{\alpha+1} \\
&-(1-\rho)\left[\lambda(y_\varepsilon) + \frac{1+\beta}{\beta}\hat{C}\langle y_\varepsilon\rangle^m\right] \leq \omega(\varepsilon) + \omega(\frac{\iota}{\varepsilon^2}).
\end{aligned}$$

Letting first ι go to 0 and then sending ε to 0, we finally conclude the desired viscosity subsolution property of w.

Step 2: smooth strict supersolution. We are now going to construct smooth strict supersolutions to (A.4) on $[T-\tau, T)$ for some small $\tau > 0$. To this end, let

$$\psi(t,y) := (1-\rho)C\langle y\rangle^m e^{L(T-t)}$$

where $L, C > 0$ will be chosen later.

Since $\lambda, \phi \in C_m(\mathbb{R}^d)$ and $u \in \mathcal{SSG}_m^-([0,T]\times\mathbb{R}^d)$, we choose a large enough constant $\bar{C}$ such that for $\zeta = \lambda, \phi$

$$\zeta(y) \le \bar{C}\langle y\rangle^m, \quad y \in \mathbb{R}^d,$$

and such that

$$u(t,y) \le \bar{C}\langle y\rangle^m, \quad (t,y) \in [0,T]\times\mathbb{R}^d. \tag{A.7}$$

Note that

$$D\langle y\rangle^m = m\langle y\rangle^{m-2}y, \quad D^2\langle y\rangle^m = m\langle y\rangle^{m-4}\left(\langle y\rangle^2 I + (m-2)y\otimes y\right).$$

Since b, σ grow at most linearly,

$$\begin{aligned}
\mathcal{L}\psi(t,y) &\le (1-\rho)Ce^{L(T-t)}\left[\bar{C}(1+|y|)|D\langle y\rangle^m| + \bar{C}^2(1+|y|)^2|D^2\langle y\rangle^m|\right]\\
&\le (1-\rho)Ce^{L(T-t)}\left[2m\bar{C}\langle y\rangle^m + 2m(m-1)\bar{C}^2\langle y\rangle^m\right]\\
&\le [2m\bar{C} + 2m(m-1)\bar{C}^2]\psi(t,y).
\end{aligned}$$

Recalling that $(m-1)(\alpha+1) = m$, we have

$$\begin{aligned}
&(\frac{1-\rho}{2})^{-\alpha}\bar{C}^{\alpha+1}|D\psi(t,y)|^{\alpha+1}\\
=&(\frac{1-\rho}{2})^{-\alpha}\bar{C}^{\alpha+1}\cdot(1-\rho)^{\alpha+1}C^{\alpha+1}e^{(\alpha+1)L(T-t)}|D\langle y\rangle^m|^{\alpha+1}\\
\le&[2^\alpha m^{\alpha+1}\bar{C}^{\alpha+1}C^\alpha e^{\alpha L(T-t)}]\psi(t,y)
\end{aligned}$$

By condition (F.4),

$$(1-\rho)\left[\lambda(y) + \frac{1+\beta}{\beta}\hat{C}\langle y\rangle^m\right] \le (1-\rho)\frac{1+2\beta}{\beta}\bar{C}\langle y\rangle^m \le \frac{1+2\beta}{\beta}\frac{\bar{C}}{C}\psi(t,y)$$

Choosing $C > \max\{2m\bar{C} + 2m(m-1)\bar{C}^2, 2^\alpha m^{\alpha+1}\bar{C}^{\alpha+1}, \frac{1+2\beta}{\beta}\bar{C}\}$, we have

$$\begin{aligned}
&-\partial_t\psi(t,y) - \mathcal{L}\psi(t,y) - (\frac{1-\rho}{2})^{-\alpha}\bar{C}^{\alpha+1}|D\psi(t,y)|^{\alpha+1}\\
&-(1-\rho)\left[\lambda(y) + \frac{1+\beta}{\beta}\hat{C}\langle y\rangle^m\right]\\
>&\psi(t,y)\left[L - C - 1 - C^{\alpha+1}e^{\alpha L(T-t)}\right].
\end{aligned}$$

Then taking $L > C + 1 + C^{\alpha+1}e$, we get

$$\begin{aligned}
&-\partial_t\psi(t,y) - \mathcal{L}\psi(t,y) - (\frac{1-\rho}{2})^{-\alpha}\bar{C}^{\alpha+1}|D\psi(t,y)|^{\alpha+1}\\
&-(1-\rho)\left[\lambda(y) + \frac{1+\beta}{\beta}\hat{C}\langle y\rangle^m\right] > 0
\end{aligned}$$

for all $y \in \mathbb{R}^d$ and $t \in [T-\tau, T)$, where $\tau = \frac{1}{\alpha L}$.

STEP 3: CONCLUSIONS. Since $w \in USC([T-\tau, T] \times \mathbb{R}^d) \cap \mathcal{SSG}_m^-$, the function $w - \psi$ attains its maximum at some point $(t, y) \in [T-\tau, T] \times \mathbb{R}^d$. We claim that $t = T$. Indeed, suppose to the contrary that $t < T$. Then, since w is a viscosity subsolution of (A.4), by taking ψ as a test function,

$$\begin{aligned}
&-\partial_t \psi(t,y) - \mathcal{L}\psi(t,y) - (\frac{1-\rho}{2})^{-\alpha} \bar{C}^{\alpha+1} |D\psi(t,y)|^{\alpha+1} \\
&- (1-\rho)\left[\lambda(y) + \frac{1+\beta}{\beta}\hat{C}\langle y\rangle^m\right] \leq 0.
\end{aligned}$$

This contradicts the fact that ψ is a strict supersolution to (A.4). Therefore, for all $(t, y) \in [T-\tau, T] \times \mathbb{R}^d$,

$$w(t,y) - \psi(t,y) \leq w(T,y) - \psi(T,y) \leq (1-\rho)\phi(y) - (1-\rho)C\langle y\rangle^m \leq 0$$

where the last inequality follows from $C > \bar{C}$. In particular, $w(t, y) \leq \psi(t, y)$. Letting $\rho \to 1$, we get $u \leq v$ on $[T-\tau, T] \times \mathbb{R}^d$.

The preceding argument can be iterated on time intervals of the same length τ. Indeed, let us choose C, L, τ as in Step 2 and put

$$\psi(t,y) := (1-\rho)C\langle y\rangle^m e^{L(T-\tau-t)}$$

on $[T-2\tau, T-\tau]$. It follows by (A.7) and the previously established inequality $u \leq v$ on $[T-\tau, T] \times \mathbb{R}^d$ that for all $y \in \mathbb{R}^d$,

$$w(T-\tau, y) = u(T-\tau, y) - \tilde{v}(T-\tau, y) \leq (1-\rho)u(T-\tau, y) \leq (1-\rho)\bar{C}\langle y\rangle^m.$$

Following the same arguments as above, we have for all $(t, y) \in [T-2\tau, T-\tau] \times \mathbb{R}^d$ that

$$w(t,y) - \psi(t,y) \leq w(T-\tau, y) - \psi(T-\tau, y) \leq (1-\rho)\bar{C}\langle y\rangle^m - (1-\rho)C\langle y\rangle^m \leq 0.$$

These arguments can be iterated to complete the proof. □

Remark A.3.3. It is worth noting that the constant $\hat{C}$ in (A.4) is exactly derived from the upper bound of v in (A.3) when estimating $\tilde{F}$ in (A.6). We show below that using the constant derived from the upper bound of u instead is also feasible. To this end, we estimate $\tilde{F}$ in the following way:

$$\begin{aligned}
\tilde{F} &= \rho F(x_\varepsilon, \frac{\tilde{v}}{\rho}) - F(x_\varepsilon, u) + \rho F(y_\varepsilon, \frac{\tilde{v}}{\rho}) - \rho F(x_\varepsilon, \frac{\tilde{v}}{\rho}) \\
&\geq (\rho - 1)\lambda(x_\varepsilon) + \frac{|u|^{\beta+1}}{\beta\eta(x_\varepsilon)^\beta} - \rho^{-\beta}\frac{|\tilde{v}|^{\beta+1}}{\beta\eta(x_\varepsilon)^\beta} - \omega_R(|x_\varepsilon - y_\varepsilon|) \\
&\geq (\rho - 1)\lambda(x_\varepsilon) + (1-\rho^{-\beta})\frac{|u|^{\beta+1}}{\beta\eta(x_\varepsilon)^\beta} + \rho^{-\beta}(\frac{|u|^{\beta+1}}{\beta\eta(x_\varepsilon)^\beta} - \frac{|\tilde{v}|^{\beta+1}}{\beta\eta(x_\varepsilon)^\beta}) \\
&\quad - \omega_R(|x_\varepsilon - y_\varepsilon|) \\
&\geq -(1-\rho)\lambda(x_\varepsilon) - 2^\beta(1-\rho)\hat{C}\langle x_\varepsilon\rangle^m - \omega_R(|x_\varepsilon - y_\varepsilon|),
\end{aligned} \tag{A.8}$$

In the last inequality we used the facts that $u^{\beta+1}(t,y) \le \hat{C}\eta^{\beta}(y)\langle y\rangle^m$ on $[0,T]\times\mathbb{R}^d$ and $\rho^{-\beta}-1 \le 2^{\beta}(1-\rho)$ for $\rho\in(\frac{1}{2},1)$.

The next lemma establishes a comparison principle for continuous solutions to (4.8) when imposed with a singular terminal time. The proof uses the shifting argument given in [GHS18].

Lemma A.3.4. *Assume that (L.1)-(L.3), (F.1) and (F.2) hold. Let $\tilde{m}$ be as in condition (F.1). Let $\underline{v}, \overline{v} \in C_{\tilde{m}}([0,T^-]\times\mathbb{R}^d)$ be a nonnegative viscosity sub- and a nonnegative viscosity supersolution to* (4.8)*, respectively, such that*

$$\lim_{t\to T}\overline{v}(t,y) = +\infty \quad \textit{locally uniformly on } \mathbb{R}^d.$$

Then,

$$\underline{v} \le \overline{v} \qquad \textit{in} \quad [0,T)\times\mathbb{R}^d.$$

In particular, there exists at most one nonnegative viscosity solution to (4.8) *in $C_{\tilde{m}}([0,T^-]\times\mathbb{R}^d)$.*

Proof. Due to the time-homogeneity of the PDE in (4.8), viscosity (super-/sub-)solutions stay viscosity (super-/sub-)solutions when shifted in time. For any $\delta>0$, we define the difference function $w:[0,T-\delta)\times\mathbb{R}^d\to\mathbb{R}$ by

$$w(t,y) := \underline{v}(t,y) - \rho\overline{v}(t+\delta,y).$$

Under assumptions (F.1) and (F.2), we have that $\underline{v}$, $-\overline{v}$ belong to $\mathcal{SSG}_m^-$ and satisfy the condition (A.3) in Proposition A.3.2 on $[0,T)\times\mathbb{R}^d$. Hence, we can use the similar argument as in the proof of Proposition A.3.2 to obtain that w is a viscosity subsolution of the following PDE:

$$-\partial_t u(t,y) - \mathcal{L}u(t,y) - (\frac{1-\rho}{2})^{-\alpha}\bar{C}^{\alpha+1}|Du|^{\alpha+1} - (1-\rho)\left[\lambda(\bar{y}) + \frac{1+\beta}{\beta}\hat{C}\langle y\rangle^m\right] = 0,$$

for $(t,y)\in[0,T-\delta)\times\mathbb{R}^d\cap\{w>0\}$ and $\lim\limits_{t\to T-\delta} w(t,y) \le (1-\rho)\underline{v}(T-\delta,y)$ for $y\in\mathbb{R}^d$. In fact, Remark A.3.3 shows that we can get around the difficulty of the singularity of $\overline{v}(\cdot+\delta,\cdot)$ at time $t=T-\delta$ in this step. Following Steps 2 and 3 in the proof of Proposition A.3.2, we have that $\underline{v}(t,y)\le\overline{v}(t+\delta,y)$ on $[0,T-\delta]\times\mathbb{R}^d$. Finally, by letting $\delta\to 0$ we conclude that $\underline{v}\le\overline{v}$ on $[0,T)\times\mathbb{R}^d$ by continuity of $\overline{v}$. □

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Declaration of Independent work

I declare that I have completed the thesis independently using only the aids and tools specified. I have not applied for a doctor's degree in the doctoral subject elsewhere and do not hold a corresponding doctor's degree. I have taken due note of the Faculty of Mathematics and Natural Sciences PhD Regulations, published in the Official Gazette of Humboldt-Universität zu Berlin no. 42/2018 on 11/07/2018.

Berlin, October 15, 2019 Xiaonyu Xia